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ON THE BICENTRALIZERS OF VON NEUMANN ALGEBRAS

SANG OG KIM

1. Introduction

Connes [2] showed that if M is an injective σ-finite factor of type III_1 and $B_\phi = \boldsymbol{C}1$ for some normal faithful state ϕ on M then M is isomorphic to the Araki-wood factor. In [7], Haagerup has succeeded to show that if M is an injective factor of type III_1 with separable predual, then $B_\phi = \boldsymbol{C}1$ for every normal faithful state on M. Since injective factors of type III_λ, $0 \leqq \lambda < 1$, were classified [9], this together classifies all injective factors of type III with separable predual. It is not known for non-injective case. In this paper we consider some conditions under which the bicentralizers be trivial.

2. Preliminaries

Let M be a von Neumann algebra with a normal faithful state ϕ. we define the bicentralizer B_ϕ of ϕ as follows.

$$B_\phi = \{ a \in M \mid x_n a - a x_n \to 0 \ \sigma\text{-strongly whenever } (x_n) \text{ is a bounded sequence in } M \text{ such that } \|x_n \phi - \phi x_n\| \to 0 \}.$$

Recall that M is of type III_1 if M has no nonzero finite projection and the set $\{t \mid \sigma_t^\phi \text{ is inner}\}$ is equal to $\{0\}$ for every normal faithful state ϕ where σ_t^ϕ is the modular automorphism induced by ϕ. Let M be with separable predual. A normal faithful semifinite weight ψ on M is called dominant if (i) $M_\psi = \{x \in M \mid \sigma_t^\psi(x) = x, t \in \boldsymbol{R}\}$ is properly infinite, (ii) for every $\lambda \geqq 0$, $\lambda \psi(x) = \psi(uxu^*)$ for some unitary u in M.

Let N be a von Neumann algebra with a normal faithful trace τ and $(\theta_s)_{s \in \mathrm{R}}$ a continuous one-parameter group of automorphisms such that $\tau \circ \theta_s = e^{-s} \tau$ and let $M = N \times_\theta \boldsymbol{R}$ be the crossed product of N by θ. Then

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by [4], the dual weight ψ of τ is a dominant weight on M and $M_\psi = \pi_\theta(N) = N$ is a factor of type II_∞. Moreover by the relative commutant theorem we have $M_\psi' \cap M =$ center of $M_\psi = \boldsymbol{C}\mathrm{I}$.

The following is due to Haagerup [7].

LEMMA 1. *Let M be a factor of type* III_1 *with separable predual. Then the followings are equivalent.*

(1) *For every faithful dominant weight ψ on M and every x in M,*

$$\overline{conv}\{uxu^* \mid u \in \mathcal{U}(M_\psi)\} \cap \boldsymbol{C}1 \neq \phi$$

(the closure is σ-weak topology).

(2) *For every normal faithful state ϕ on M, $B_\phi = \boldsymbol{C}1$.*

(3) *$\{\phi \mid \phi$ is a normal faithful state on $M, M_\phi' \cap M = \boldsymbol{C}1\}$ is norm dense in M_*.*

If M is injective then (1) (equivalently (2), (3)) holds.

Note that the following is shown in [8]. Let N be a factor with separable predual and M the crossed product of N by a discrete outer action. Then M has the Dixmier property relative to N: for all $x \in M$,

$$\overline{conv}\{uxu^* \mid u \in \mathcal{U}(N)\} \cap \boldsymbol{C}1 \neq \phi.$$

But in the case of continuous crossed product, it is not known.

3. Bicentralizers of von Neumann algebras

PROPOSITION 1. *The condition* (1) *of Lemma* 1 *is equivalent to* (1').

(1'): *There is a faithful dominant weight ψ on M such that for every $x \in M$, $\overline{conv}\{uxu^* \mid u \in \mathcal{U}(M_\psi)\} \cap \boldsymbol{C}1 \neq \phi$.*

Proof. Suppose that there is a faithful dominant weight ψ such that (1') holds and let ϕ be an arbitrary faithful dominant weight. Then by [4], $\phi = \psi_v$ for some unitary v in M, where $\psi_v(x) = \psi(v^*xv)$ for $x \geqq 0$. Note that

$$\begin{aligned} \mathfrak{N}_{\psi_v} &= \{x \mid \psi_v(x^*x) < \infty\} \\ &= \{x \mid \psi(v^*x^*xv) < \infty\} \\ &= \{x \mid xv \in \mathfrak{N}_\psi\} \\ &= \mathfrak{N}_\psi \cdot v^*. \end{aligned}$$

Also, $\mathfrak{M}_{\psi_v} = (\mathfrak{N}_\psi \cdot v^*)^* (\mathfrak{N}_\psi \cdot v^*) = v\mathfrak{M}_\psi v^*$.

Hence, $M_{\psi_v} = \{y \mid \psi_v(yx) = \psi_v(xy),$ for all $x \in \mathfrak{M}_{\psi_v}\}$

$= \{y \mid \phi(v^* yxv) = \phi(v^* xyv)\}$
$= \{y \mid \phi(v^* yvz) = \phi(zv^* yv), \text{ for all } z \in \mathfrak{M}_\phi\}$
$= \{y \mid v^* yv \in M_\phi\} = vM_\phi v^*.$

Therefore, $\overline{conv}\{uxu^* \mid u \in \mathcal{U}(M_{\phi_v})\} \cap \boldsymbol{C}1 \neq \phi$.
This completes the proof.

PROPOSITION 2. *The condition* (3) *of Lemma* 1 *is equivalent to* (3′).
(3′): *There is a normal faithful state* ϕ *on* M *such that* $M_\phi' \cap M = \boldsymbol{C}1$.

Proof. Let ϕ be a normal faithful state on M then by [3], there exists a unitary u in M such that $\|\phi - \phi_u\| < \varepsilon$ This entails the proof.

Let $\mathcal{S}_n(M)$ be the set of all normal states on M and let us put $M_\phi^{asympt} = \{(x_n) \in \mathcal{L}^\infty(\boldsymbol{N}, M) \mid \|x_n\phi - \phi x_n\| \to 0 \text{ as } n \to \infty\}$ for every $\phi \in \mathcal{S}_n(M)$. Let $\|x\|_\phi = \phi(x^* x)^{1/2}$ for $x \in M$. It is known

$$B_\phi = \{a \in M \mid \lim \|u_n a - a u_n\| = 0 \text{ for all } (u_n) \in \mathcal{U}(M_\phi^{asympt})\}.$$

The author thanks Haagerup for pointing out the following fact.

PROPOSITION 3. *There exists no type* III_1 *factor with separable predual such that* M_ϕ^{asympt} *consists only of trivial central sequences for all* $\phi \in \mathcal{S}_n(M)$.

Proof. If ϕ is any normal faithful state on a factor of a type III_1, then by [3], there exists a state ϕ' arbitrary close to ϕ such that each of the centralizer $M_{\phi'}$ contains a copy of 2×2 matrices. This implies that for every $n \in N$, there exists a projection $p_n \in M$ such that

$$\|p_n\phi - \phi p_n\| < 1/n \text{ and } |\phi(p_n) - 1/2| < 1/n.$$

Clearly $(p_n) \in M_\phi^{asympt}$, but (p_n) is not a trivial central sequence because

$$\|p_n - \phi(p_n)\|_\phi^2 = \phi(p_n) - \phi(p_n)^2 \longrightarrow 1/4 \text{ for } n \to \infty$$

and because $\phi(p_n)1$ is the central element in M which is closest to p_n in $\|\cdot\|_p$-norm. This completes the proof.

PROPOSITION 4. *If the unitary group* $\mathcal{U}(M_\phi)$ *of* M_ϕ *is amenable, then* (1) *holds.*

Proof. Let m be an invariant mean on $\mathcal{U}(M_\phi)$, then for every $x \in M$,

$$y = \int_{\mathcal{U}(M\phi)} uxu^* dm(u) \in M_\phi' \cap M = \boldsymbol{C}1.$$

This completes the proof.

COROLLARY 5. *If M_ϕ is generated by an amenable subgroup of $\mathcal{U}(M_\phi)$, then* (1) *holds.*

PROPOSITION 6. *If M_ϕ is injective, then* (1) *holds.*

Proof. Since M_ϕ is an injective II_∞-factor, we can choose an increasing sequence of finite dimensional subfactor N_i of M_ϕ such that $\bigcup_{i=1}^{\infty} N_i$ is dense in M_ϕ. For $x \in M$, put $x_i = \int_{\mathcal{U}(N_i)} uxu^* d\mu_i$, where μ_i is the normalized Haar measure on $\mathcal{U}(N_i)$. Since $\{N_i\}$ is increasing, let $\lim x_i = x_\infty$ in the σ-weak topology. Then

$$x_\infty \in \bigcap_{i=1}^{\infty} (N_i' \cap M) = M_\phi' \cap M = \boldsymbol{C}1.$$

By construction, $x_\infty \in \overline{conv}\{uxu^* \mid u \in \mathcal{U}(M_\phi)\} \cap \boldsymbol{C}1$. This completes the proof.

Let M be a von Neumann algebra acting on a separable Hilbert space H and denote $\tilde{M}$ the von Neumann algebra $M \otimes M$ acting on $H \otimes H$ and by $\sigma_M \in \mathrm{Aut}(\tilde{M})$ the flip automorphism determined by $\sigma_M(a \otimes b) = b \otimes a$, $a, b \in M$.

DEFINITION. *Let $A \subset B$ be von Neumann algebras and $\alpha \in Aut(A)$ an automorphism of A. We shall say that α is B-inner if there is a unitary $u \in B$ implementing α.*

PROPOSITION 7. *If the flip automorphism $\sigma_{M\phi} \in Aut(M_\phi)$ is $\tilde{M}$-inner, then the map $a \otimes b' \longrightarrow ab'$, $a \in M$, $b' \in M'$, extends to an isomorphism of $M_\phi \otimes M'$ onto $M_\phi \vee M'$, the von Neumann algebra generated by M_ϕ and M'.*

Proof. Let $v \in \tilde{M}$ be a unitary such that $\sigma_{M\phi}(a) = v^* av$, $v \in \tilde{M}_\phi$. Define a map $\rho : M_\phi \vee M' \rightarrow v^*(M_\phi \otimes M')v$ by

$$\rho(t) = v^*(1 \otimes t)v, \quad t \in M_\phi \vee M'.$$

Then ρ is an isomorphism since it is a composition of amplication and spatial isomorphism. Now for every $a \in M_\phi$, $b' \in M'$ we have

$$\begin{aligned}\rho(ab') &= v^*(1 \otimes ab')v = v^*(1 \otimes a)vv^*(1 \otimes b')v \\ &= \sigma_{M\phi}(1 \otimes a)(1 \otimes b') = a \otimes b'.\end{aligned}$$

Hence ρ^{-1} is the desired isomorphism. This completes the proof.

PROPOSITION 8. *If* $\sigma_{M\psi} \in Aut(\tilde{M}_\psi)$ *is* $\tilde{M}$*-inner, then there exists a type* I *factor* F *such that* $M_\psi \subset F \subset M$.

Proof. By proposition 7, there is an isomorphism η from $M_\psi \vee M'$ onto $M_\psi \otimes M'$ such that $\eta(ab') = a \otimes b'$ for $a \in M_\psi$, $b \in M$. Since isomorphisms between type III-factors are always spatial, η is a spatial isomorphism. Let u be a unitary implementing η and let $F = u(B(H) \otimes 1)u^*$. Then $M_\psi \subset F \subset M$ and η is a type I factor. This completes the proof.

PROPOSITION 9. *If* $\sigma_{M\psi} \in Aut(\tilde{M}_\psi)$ *is* $\tilde{M}$*-inner, then* (1) *holds.*

Proof. By proposition 8, there exists a type I factor such that $M_\psi \subset F \subset M$. Since type I von Neumann algebras are injective by [6], F is injective.

Let $F = \overline{\bigcup_{i=1}^{\infty} N_i}$ and for $x \in M$, let $y_i = \int_{\mathcal{U}(N_i) \cap M\psi} uxu^* d\mu(u)$, where μ is the normalized Haar measure on $\mathcal{U}(N_i) \cap M_\psi$. Put $y_\infty = \lim y_i$. Then $y_\infty \in F' \cap N_i' \cap M \subset M_\psi' \cap M = \boldsymbol{C}1$. By construction, $y_\infty \in \overline{conv}\{uxu^* \mid u \in \mathcal{U}(M_\psi)\}$. This completes the proof.

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Hallym University
Chuncheon 200, Korea