Bull. Korean Math. Soc. 23(1986), No. 2, pp. 113-116

ON STRONGLY STARLIKE FUNCTIONS OF ORDER α

M. M. ELHOSH

1. Introduction

Let $S^*(\alpha)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

that are analytic and satisfy the condition

$$|\arg zf'(z)/f(z)|\leq\pi\alpha/2,\quad (0\leq\alpha\leq 1), \tag{2}$$

in $|z|<1$. Then $f(z)$ is said to be strongly starlike of order α [3]. Functions in $S^*(\alpha)$ are univalent, since for each α, $0\leq\alpha\leq 1$, $S^*(\alpha)$ is a subclass of the class of close-to-convex functions of order α $(0\leq\alpha\leq 1)$ and $S^*(1)$ is the ordinary class of starlike functions (see [3], [12]).

Let also $f(z)$ be of the form (1) and satisfy

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n z^n \tag{3}$$

in $|z|<1$. Then this is called the logarithmic expansion of $f(z)$ in $|z|<1$ and γ_n are the logarithmic coefficients (see for example [6], [7])

In this paper we shall use arguments of [2], [4] to obtain integral means for the logarithmic derivatives. Then we use these and the technique of [8] to obtain coefficient difference bounds.

2. The logarithmic derivatives

THEOREM 1. *Let $f\in S^*(\alpha)$. Then for $z=re^{i\theta}(0<r<1)$, $0\leq\alpha\leq 1$, $-\infty<p<\infty$ and $K(z)=z(1-z)^{-2}$ we have*

$$\int_{-\pi}^{\pi}\left|r\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{p}d\theta\leq\int_{-\pi}^{\pi}\left|r\frac{K'(re^{i\theta})}{K(re^{i\theta})}\right|^{p\alpha}d\theta.$$

In particular, if $p=2$ then

$$\int_{0}^{2\pi}\left|r\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{2}d\theta\leq\int_{0}^{2\pi}\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^{2\alpha}d\theta$$

Received January 7, 1985.

$$=0(1)\begin{cases}(1-r)^{1-2\alpha}, & (\alpha>1/2),\\ \log(1/(1-r)), & (\alpha=1/2)\\ 1, & (0\leq\alpha<1/2),\end{cases}$$

and this gives a partial answer to questions raised in [1, *Problem* 6.43] *and* [5, *Problem* 6.71].

Proof of Theorem 1. We see from [3], [12] that (2) can be written in the form

$$zf'(z)/f(z)=p(z)^{\alpha} \tag{4}$$

where $p(z)=1+p_1z+\ldots$ such that Re $p(z)>0$ in $|z|<1$.
Now in view of [2], [4] we have

$$\left[\pm\log\left|\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right|\right]^{*}\leq\alpha\left[\log\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right]^{*}$$
$$=\alpha\left[\pm\log\left|\frac{re^{i\theta}K'(re^{i\theta})}{K(re^{i\theta})}\right|\right]^{*} \tag{5}$$

by (4).

This gives that

$$\int_{-\pi}^{\pi}\phi\left(\pm\log\left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|\right)d\theta\leq\int_{-\pi}^{\pi}\phi\left[\pm\log\left|\frac{rK'(re^{i\theta})}{K(re^{i\theta})}\right|^{\alpha}\right]d\theta.$$

Theorem 1 now follows by using the equation $\phi(u)=e^{pu}$.

REMARK 1. Using (5) and the argument of [2, p. 346] we see that

$$(\pm\log|f'(re^{i\theta})|)^{*}\leq\left[\pm\log\frac{|1+re^{i\theta}|^{\alpha}}{|1-re^{i\theta}|^{\alpha+2}}\right]^{*}$$

which is Brown's result [4] when $0\leq\beta\leq1$ for our particular class.

THEOREM 2. *Let* $f\in S^{*}(\alpha)$ *and that* (3) *holds; then for* $n\geq1$ *we have*

$$|\gamma_n|\leq\alpha/n.$$

The function $f(z)=[(1+z^n)/(1-z^n)]^{\alpha}$ *shows that this bound is the best possible.*

Proof. Let

$$zf'(z)/f(z)=p(z)^{\alpha}=1+\sum_{n=1}^{\infty}b_n(\alpha)z^n.$$

Then, from [12, p. 459] we have

$$|b_n(\alpha)|\leq2\alpha\quad(n\geq1).$$

Using this and (3), (4), we easily see that

$$|\gamma_n|\leq\alpha/n,\quad(0\leq\alpha\leq1,n\geq1), \tag{6}$$

as required.

From (6) one can also see that Milin's inequalities [1, p. 141] and [2, p. 335] for ourclass are

$$\sum_{k=1}^{n} k|\gamma_k|^2 \leq \sum_{k=1}^{n} \alpha^2/k, \tag{7}$$

$$\sum_{m=1}^{n} \sum_{k=1}^{m} k|\gamma_k|^2 \leq \sum_{m=1}^{n} \sum_{k=1}^{m} \alpha^2/k. \tag{8}$$

3. Coefficient bounds

Applying Theorem 2, we deduce

THEOREM 3. *Let* $f \in S^*(\alpha)$ *and* $f(z) = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}$ *in* $|z| < 1$. *Then we have for* $k \geqq 1$ *that*

$$|a_{nk+1}| \leq \binom{n-1+\alpha/k}{n},$$

$$\big||a_{nk+1}| - |a_{(n-1)k+1}|\big| \leq A(\alpha, k) n^{-1+1/k},$$

where $A(\alpha, k)$ *is a constant depending on* α, k.

Proof. The case $\alpha = 1$ can be found in [8] and [11]. For $0 < \alpha < 1$ we have by Theorem 2 that $\log f(z)/z \langle\langle (2\alpha/k) \log(1/(1-z^k))$ where, as in [9, p. 52], $\langle\langle$ means the coefficients on the left are dominated by the coefficients on the right of $\langle\langle$.

Now since exponentiation preserves majorization we have

$$f(z)/z \langle\langle (1-z^k)^{-2\alpha/k}$$

which gives the first inequality of Theorem 3.

Finally, applying the coefficients formula on

$$(z^k - z_1^k) f'(z) = -z_1^k + \sum_{n=1}^{\infty} [(nk+1) a_{nk+1} - ((n-1)k+1) a_{(n-1)k+1}] z^{nk}$$

and using Goluzin's inequality $|z^k - z_1^k| \, |f(z)| \leq A(k)(1-r^k)^{-1/k}$ and the fact that $\int_0^{2\pi} |p(re^{i\theta})|^\alpha d\theta \leq A(\alpha, k)$ for $0 < \alpha < 1$, we deduce the second inequality (just as in [8]) of Theorem 3.

REMARK. Setting $\alpha = 1$ and $k = 2$ in the second inequality of Theorem 3 we deduce Milin's estimate [10] which gives a partial answer to the question raised in [1, prob. 6. 37].

References

1. J. M. Anderson, K. F. Barth and D. A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. **9**(1977), 129-162.
2. A. Baernstein and J. Brown, *Integral means of derivatives of monotone slit mappings*, Comment. Math. Helvetici, **57**(1982), 331-348.
3. D. Brannan and W. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. **2**(1969), 431-443.
4. J. Brown, *Derivatives of close-to-convex functions, integral means and bounded mean oscillation*, Math. Z. 178(1981), 353-358.
5. J. Clunie and D. Brannan, *Aspects of Contemporary Analysis*, Academic Press (1980).
6. P. Duren, *Successive coefficients of univalent functions*, J. Lond. Math. Soc. **2**(19) (1979), 448-450.
7. P. Duren and Y. Leung, *Logarithmic coefficients of univalent functions*, J. d'Analyse **36**(1979), 36-43.
8. M. M. Elhosh, *On successive coefficients of close-to-convex functions*, Proc. Roy. Soc. Edinb. 96A(1984), 47-49.
9. E. Hille, *Ordinary Differential Equations in the Complex Domain* (John Wiley 1976).
10. I. M. Milin, *Adjacent coefficients of odd univalent functions*, Siberian Math. J. (English version), **22**(1981), 283-290.
11. Ch. Pommerenke, *On the coefficients of close-to-convex functions*, Michigan Math. J. **9**(1962), 259-269.
12. M. O. Reade, *The coefficients of close-to-convex functions*. Duke Math. J. **23**(1956), 459-462.

Department of Pure Mathematics
The University College of Wales
Aberystwyth
Dyfed
Wales
UK