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A NOTE ON FUNCTION SPACES

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1. Introduction and preliminaries

For the topological spaces X and Y, let Y^X be the family of all continuous functions from X to Y with the compact-open topology, and let w be the evaluation map of Y^X.

In this note, we will give an example in which the graph of the evaluation map of Y^X is not closed (See [1], problem 3 in section 1 in p. 276.). And also, in case of $Y=I$ (the closed unit interval), we will give an example in which the evaluation map of I^X is continuous, but X is not locally compact (See [1], example 1 in p. 260.).

Throughout this note, all topological spaces are assumed to be Hausdorff.

DEFINITION 1 [1]. *For each pair of sets $A\subset X$, $B\subset Y$, let* $(A, B)=\{f\in Y^X | f(A)\subset B\}$. *The compact-open topology in Y^X is that having as sub-basis all sets (A, V), where $A\subset X$ is compact and $V\subset Y$ is open.*

DEFINITION 2 [1]. *For any two topological spaces X, Y, the map $w: Y^X\times X\to Y$ defined by $w(f, x)=f(x)$ is called the evaluation map of Y^X.*

THEOREM 1 [1]. *If X is locally compact, then the evaluation map $w: Y^X\times X\to Y$ is continuous.*

For a map f from X to Y, the subset $\{(x, f(x)) | x\in X\}$ of $X\times Y$ is called the graph of f and denoted by $G(f)$.

THEOREM 2 [2]. *If f is a continuous function from X to Y, then $G(f)$ is closed.*

THEOREM 3 [2]. *Let $f:X\to Y$ be a function. If Y is compact and $G(f)$ is closed, then f is continuous.*

2. Main results

The following example is given in [3], as an example in which the evaluation map of Y^X is not continuous. Furthermore, we will show that the example is a counter-example for the problem, "Let Y be Hausdorff. If the compact-open topology is used in Y^X, then $G(w)=\{(f,x,y)|f(x)=y\}\subset Y^X\times X\times Y$ is closed."

EXAMPLE 1. Let $Y=R$(set of all real numbers) with the usual topology and $X=Q$(set of all rational numbers) with the relative topology of R. We will show that $G(w)=\{(f,x,y)|f(x)=y\}$ is not closed. Suppose that $G(w)$ is closed, and let $j:Q\to R$ be the inclusion map, then $j\in R^Q$ since j is continuous. Since $j(0)\neq 1$, we have $(j,0,1)\notin G(w)$. By the closedness of $G(w)$, there exist a basic open neighborhood $\bigcap_{i=1}^{n}(K_i,W_i)$ of j, where K_i is compact in Q and W_i is open in R, an open neighborhood U of 0 in Q and an open neighborhood W of 1 in R such that $(j,0,1)\in\bigcap_{i=1}^{n}(K_i,W_i)\times U\times W$ and $\bigcap_{i=1}^{n}(K_i,W_i)\times U\times W\cap G(w)=\phi$. Since $j\in\bigcap_{i=1}^{n}(K_i,W_i)$, we have $K_i=j(K_i)\subset W_i$ for each $i=1,2,\cdots,n$. If $K=\bigcup_{i=1}^{n}K_i\supset U$, then K is a compact neighborhood of 0 in Q. But 0 has no compact neighborhood in Q, and so $K\not\supset U$. Hence, there exists an $x\in U-K$. By the closedness of K, there exists a basic open set $B(x,\delta)=\{y\in Q \mid |y-x|<\delta\}$ such that $B(x,\delta)\cap K=\phi$.

Define $f:Q\to R$ given by

$$f(r)=\begin{cases} 1+\dfrac{1-x+\delta}{\delta}(r-x) \text{ if } r\in B(x,\delta) \text{ and } r\leq x, \\ 1+\dfrac{x-1+\delta}{\delta}(r-x) \text{ if } r\in B(x,\delta) \text{ and } r>x, \\ r \text{ if } r\notin B(x,\delta). \end{cases}$$

Then $f\in R^Q$, and for each $i=1,2,\cdots,n$, we have $f(K_i)=K_i$ since $B(x,\delta)\cap K_i=\phi$ for any $i=2,\cdots,n$ and $f|_{Q-B(x,\delta)}=j$. Since $K_i\subset W_i$, we have $f\in\bigcap_{i=1}^{n}(K_i,W_i)$, and so, $(f,x,1)\in\bigcap_{i=1}^{n}(K_i,W_i)\times U\times W$.

But $(f,x,1)\in G(w)$ since $f(x)=1$. This is impossible, so $G(w)$ is not closed.

Next, in order to give a counter-example for the "only if" part of the remark, "$w:I^X\times X\to I$ is continuous if and only if X is locally compact."(the "if" part of the remark is obviously true by Theorem 1), the following Theorem is necessary.

THEOREM 4. *If Y is compact, then $w:Y^X\times X\to Y$ is continuous if and only if $G(w)$ is closed.*

PROOF: Obvious by Theorem 2 and Theorem 3.

EXAMPLE 2. Let $X=\{(x,y)\mid y\geq 0,\ x,y\in Q\}$ and fix some irrational number θ. Topologize on X with the topology generated by ε-neighborhoods of the form $N_\varepsilon((x,y))=\{(x,y)\}\cup B_\varepsilon\left(x+\frac{y}{\theta}\right)\cup B_\varepsilon\left(x-\frac{y}{\theta}\right)$ where $B_\varepsilon(t)=\{r\in Q \mid |r-t|<\varepsilon\}$, Q being the rationals on the x-axis. This topology is called "Irrational Slope Topology (see[4])". Each $N_\varepsilon((x,y))$ consists of $\{(x,y)\}$ plus two intervals on the rational x-axis centered at the two irrational points $x \pm y/\theta$; the lines joining these points to (x,y) have slope $\pm\theta$.

For the topological space X, the followings are easily shown in [4].

1) X is Hausdorff.

2) X is not completely regular.

3) Every real-valued continuous function on X is constant. By 2), we have that X is not locally compact. If we show that the graph of the evaluation map of I^X is closed, then the evaluation map of I^X is continuous by Theorem 4.

Let $(f,x,t) \notin G(w)$, then $f(x) \neq t$. Hence, there exist the open neighborhoods U, V of $f(x)$, t, respectively, such that $U \cap V = \phi$, so $(\{x\}, U) \times X \times V$ is an open neighborhood of (f,x,t). And also, if $(g,x',t') \in (\{x\}, U) \times X \times V$, then $g(x) \in U$, $x' \in X$ and $t' \in V$. By $U \cap V = \phi$ and 3), we have $g(x') = g(x) \neq t'$, therefore $(g,x',t') \notin G(w)$. Hence, $G(w)$ is closed.

The following is obtained under the hypothesis that X is completely regular.

THEOREM 2. *If X is completely regular, then the evaluation map $w: R^X \times X \rightarrow R$ is continuous if and only if X is locally compact.*

PROOF: Only the sufficiency requires proof. Suppose that X is not locally compact, then there exists an $x \in X$ which has no compact neighborhood. Define $f: X \rightarrow R$ given by $f(x)=0$ for any $x \in X$, and let $W = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Since w is continuous at (f,x) and $w(f,x) = f(x) = 0$, there exist a neighborhood $\bigcap_{i=1}^{n} (K_i, U_i)$ of f and a neighborhood V of x

such that $w(\bigcap_{i=1}^{n}(K_i, U_i)\times V)\subset W$. Since x has no locally compact neighborhood, we have $\bigcup_{i=1}^{n} K_i \not\supset V$. Hence $V-\bigcup_{i=1}^{n} K_i \neq \phi$, and let $y\in V-\bigcup_{i=1}^{n} K_i$. Since X is completely regular and $\bigcup_{i=1}^{n} K_i$ is closed in X, there exists a $g\in R^X$ such that $g(\bigcup_{i=1}^{n} K_i)=\{0\}$ and $g(y)=1$. Hence $g(\bigcup_{i=1}^{n} K_i)=\{0\}\subset U_i$ for any $i=1, 2, \cdots, n$, and so $g\in\bigcap_{i=1}^{n}(K_i, U_i)$ and $y\in V$, but $w(g, y)=g(y)=1\not\in W$. This is impossible, and so X is locally compact.

REMARK. In the above theorem, the complete regularity of X is essential by the example 2.

COROLLARY. *For the completely regular space X, the followings are equivalent.*

(1) *The evaluation map $w: I^X\times X\rightarrow I$ is continuous.*

(2) *The evaluation map w has the closed graph.*

(3) *X is locally compact.*

References

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