PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 2, 87—96, 1986

A NOTE ON M-HYPONORMAL OPERATORS IN HILBERT SPACE

YOUNG SIK PARK AND JE YOON LEE

1. Introduction

In this paper H is a separable, infinite dimensional complex Hilbert space with inner product $(\cdot,\cdot)$, and the Banach algebra of all bounded linear operators on H will be denoted by $L(H)$. $T\in L(H)$ is called dominant by J. Stamfli and B. Wadhwa if, for all complex λ, $\operatorname{ran}(T-\lambda)\subset \operatorname{ran}(T-\lambda)^*$ or, equivalently, if there exists a positive number M_λ such that $\|(T-\lambda)^*f\|\leq M_\lambda\|(T-\lambda)f\|$ for all f in H. If there exists a constant $M\geq 1$ such that $M_\lambda\leq M$ for all λ, T is called M-hyponormal, and if $M=1$, T is hyponormal. The purpose of the present note is to give several properties of M-hyponormal operators, and show some relations when T is of M-power class (N) or T is of class (N). Therefore we know that an M-power class is strictly larger than the class of hyponormal operators.

2. Preliminaries

LEMMA 2.1 [13]. *If T is an M-hyponormal operator, then* (1) *$Tx=zx$ implies that $T^*x=\bar{z}x$ for all $z\in C$ and* (2) *$\|(T-z)x\|^{n+1}\leq M^{\frac{n(n+1)}{2}}\|(T-z)^{n+1}x\|$ for all $z\in C$, $n=1,2,\cdots$.*

LEMMA 2.2. *T is an M-hyponormal operator if and only if $M^2(T-z)^*(T-z)-(T-z)(T-z)^*\geq 0$ for all $z\in C$.*

PROOF. If T is M-hyponormal, then there exists a positive number M such that $\|(T-z)^*x\| \leq M\|(T-z)x\|$ for all $x \in H$ and for all $z \in C$, and so $\|(T-z)^*x\|^2 = ((T-z)^*x, (T-z)^*x) \leq M^2((T-z)x, (T-z)x)$, and thus we have $M^2(T-z)^*(T-z)-(T-z)(T-z)^* \geq 0$. Conversely, if $M^2(T-z)^*(T-z)-(T-z)(T-z)^* \geq 0$, then $M^2((T-z)^*(T-z)-(T-z)(T-z)^*)x, x) \geq 0$, and so we have $\|(T-z)^*x\|^2 \leq M^2\|(T-z)x\|^2$. Thus T is an M-hyponormal operator.

LEMMA 2.3 [11]. *From Lemma 2.2 the following statements are each equivalent to each other;*

(1) *T is an M-hyponormal operator,*

(2) *For each $z \in C$, there exists an operator $A_z \in L(H)$ such that $T-z=(T-z)^*A_z$.*

LEMMA 2.4 [12, Theorem B]. *If T and T^* are M-hyponormal operators, then T is normal.*

We shall now give an example of an M-hyponormal operator which is not hyponormal.

EXAMPLE 2.5 [13]. *Let $\{e_i\}^\infty_{i=1}$ be an orthonormal basis of a Hilbert space. Let T be a weighted shift defined by $Te_1=e_2$, $Te_2=2e_3$, and $Te_i=e_{i+1}$ for $i \geq 3$. Then $T^*e_1=0$, $T^*e_2=e_1$, $T^*e_3=2e_2$ and $T^*e_i=e_{i-1}$ for $i \geq 4$.*

3. Properties of M-hyponormal operators

LEMMA 3.1 [5]. *From Lemma 2.3 an operator A_z is constructed as follows;*

(1) *$A_z^*((T-z)x)=(T-z)^*x$,* (2) *$A_z^*y=0$ for every $y \in (ran(T-z))^\perp$, and (3) $\|A_z\| \leq \lambda$.*

THEOREM 3.2. *T is an M-hyponormal operator if and only if there exists a positive contraction P such that $(T-z)(T-z)^*=P(T-z)^*(T-z)$ and P commutes with $(T-z)^*(T-z)$.*

PROOF. If T is an M-hyponormal operator, we have $((T-z)(T-z)^*)^2 \leq M^4((T-z)^*(T-z))^2$, that is, $((T-z)(T-z)^*((T-z)(T-z)^*)^* \leq (M^2)^2((T-z)^*(T-z))\ ((T-z)^*(T-z))^*$. It follows from Lemma 3.1 (1) that $(T-z)(T-z)^* =(T-z)^*(T-z)A_z$, and so we have $(T-z)(T-z)^*=A_z^*(T-z)^*(T-z)$. So we put $P=A_z^*$. Then it is clear by lemma 3.1 (2) that $(P(x_1+x_2),\ x_1+x_2)=(Px_1,x_1)+(Px_1,x_2)+(Px_2,x_1)+(Px_2,\ x_2)=(Px_1,x_1)\geq 0$ for every $x_1 \in \overline{\text{ran}(T-z)^*(T-z)}$ and $x_2 \in (\text{ran}(T-z)^*(T-z))^{\perp}$. Also, we have by Lemma 3.1 (3) that $\|A_z^*\|=\|A_z\| \leq \lambda$, and thus $P=A_z^*$ is a positive contraction. Since P is a positive operator on H, P is self-adjoint, that is, $A_z^*=A_z$. Thus we have $P(T-z)^*(T-z)=(T-z)(T-z)^*=((T-z)(T-z)^*)^*=(P(T-z)^*(T-z))^*=(T-z)^*(T-z)P$, hence P commutes with $(T-z)^*(T-z)$. Conversely, if there exists a positive contraction P such that $(T-z)(T-z)^*=P(T-z)^*(T-z)$ and P commutes with $(T-z)^*(T-z)$, then we have $((T-z)(T-z)^*)^2=P^2(T-z)^*(T-z))^2 \leq ((T-z)^*(T-z))^2 \leq M^4((T-z)^*(T-z))^2$ for positive number M. Therefore $(T-z)(T-z)^* \leq M^2(T-z)^*(T-z)$, and thus it follows from Lemma 2.2 that T is an M-hyponormal operator on H.

LEMMA 3.3 [1, Proposition 2]. *If T is a bounded linear operator such that T^*T commutes with T^*+T then $4T^*T-(T^*+T)^2\geq 0$.*

LEMMA 3.4 [11, Corollary 8]. *If T is M-hyponormal, N is normal and $TN=NT$ then $T+N$ is an M-hyponormal operator.*

THEOREM 3.5. *If T is an M-hyponormal operator such that T^*T commutes with T^*+T and $TC=CT$, where C is any one root of the equation $(z-T^*)(z-T)=0$ for $z\in$*

C, then $T+C^*$ is an M-hyponormal operator.

PROOF. If T is a bounded linear operator such that T^*T, commutes with T^*+T, it follows from Lemma 3.3 that $4T^*T-(T^*+T)^2\geq 0$. Put $C=\dfrac{(T^*+T)+i\sqrt{4T^*T-(T^*+T)^2}}{2}$

Then it is clear that C is normal, and also C^* is normal. From [3] it is obvious that $TC^*=C^*T$, hence it follows from Lemma 3.4 that $T+C^*$ is an M-hyponormal operator.

COROLLARY 3.6. *If T and T^* are M-hyponormal operators then $T+T^*$ is an M-hyponormal operator.*

PROOF. Since T and T^* are M-hyponormal operators, it follows from Lemma 2.4 that T is normal, and thus $T+T^*$ is an M-hyponormal operator.

LEMMA 3.7 [12, *Theorem* 1]. *Let T be M-hyponormal and suppose that $TX=XT^*$ for $X\in L(H)$. Then $T^*X=XT$.*

LEMMA 3.8. *If A and B are M-hyponormal and $AX=XB^*$ for $X\in L(H)$, then $A^*X=XB$.*

PROOF. From Lemma 3.7.

THEOREM 3.9. *If T and A are M-hyponormal and $TA_z=A_zA^*$ where A_z is a bounded linear operator in Lemma 3.1, then $\overline{ranA_z}$ reduces T and $Ker(T-z)$ reduces A.*

PROOF. It follows from Lemma 3.8 that $TA_zA_z^*=A_zA^*A_z^*=A_zA_z^*T$, and thus T commutes with $A_zA_z^*$. Since $A_zA_z^*$ is self-adjoint, $A_zA_z^*$ is normal, and $\overline{\text{ran}(A_zA_z^*)}=\overline{\text{ran}A_z}$. Therefore $A_zA_z^*$ is the projection on $\overline{\text{ran }A_z}$. Since T commutes with $A_zA_z^*$, $\overline{\text{ran }A_z}$ reduces T. Similarly, it is obvious by Lemma 3.8 that $A_z^*A_zA=A_z^*T^*A_z=(TA_z)^*A_z=AA_z^*A_z$, and thus A commutes with $A_z^*A_z$. Also $A_z^*A_z$ is normal and $\text{Ker}(A_z^*A_z)=\text{Ker}A_z$. Thus $A_z^*A_z$ is the projection on Ker A_z. Since A commutes with $A_z^*A_z$, Ker A_z

reduces A. From Lemma 3.1 R. G. Douglas [2] has obtained that there exists a unique bounded operator A_z such that Ker $A_z = \text{Ker}(T-z)$. Therefore $\text{Ker}(T-z)$ reduces A.

THEOREM 3.10. *Let A and B be M-hyponormal operators such that $AX = XB^*$ for $X \in L(H)$. Then A is a linear conbination of four unitary operators each of which commutes with XX^*.*

PROOF. By Lemma 3.8, A commutes with XX^* and XX^* is normal. Let $A = H + iJ$ be the Cartesian decomposition of A, where $H = \frac{1}{2}(X + X^*)$ and $J = \frac{1}{2i}(X - X^*)$. Then H and J commute with XX^*. It can assumed that H and J are contractions, and thus $H \pm i(I-H^2)^{\frac{1}{2}}$ and $iJ \pm (I-J^2)^{\frac{1}{2}}$ are unitary, commutes with XX^*. Since $2A = (H + i(I-H^2)^{\frac{1}{2}}) + (H - i(I-H^2)^{\frac{1}{2}}) + (iJ + (I-J^2)^{\frac{1}{2}}) + (iJ - (I-J^2)^{\frac{1}{2}}) = 2H + 2iJ$, $A = \text{span}\{(H + i(I-H^2)^{\frac{1}{2}}),\ (H - i(I-H^2)^{\frac{1}{2}}), (iJ + (I-J^2)^{\frac{1}{2}}), (iJ - (I-J^2)^{\frac{1}{2}})\}$.

We shall consider a class $\mathcal{L}$ of operators T satisfying the inequality $T^*T \geq (ReT)^2$. By Che-Kao Fong, Vasile I. Istratescu [10] every hyponormal operator is in $\mathcal{L}$. Now we shall show an example of an $M(=3)$-hyponormal operator T if T is in $\mathcal{L}$.

LEMMA 3.11 [10, Proposition 2.1]. *If T is in $\mathcal{L}$ and z is a real number, then $T-z$ is in $\mathcal{L}$*

EXAMPLE 3.12. Let $T \in L(H)$ be in $\mathcal{L}$. Then T is an M $(=3)$-hyponormal operator.

PROOF. If z is real number, by Lemma 3.11 it sufficient to consider the case when $z=0$. Then we have $(||T^*x|| - ||Tx||)^2 = ||T^*x||^2 + ||Tx||^2 - 2||T^*x||\ ||x|| \leq ||(T+T^*)\ x||^2 = ||2Re\ Tx||^2 - 4(Re\ T)^2(x,x) \leq 4(T^*Tx, x) = (2||Tx||)^2$, and thus $||T^*x|| \leq 3||Tx||$. Therefore it is clear that $||(T-z)^*x$

$|| \leq 3||(T-z)x||$ for all $x \in H$ and all real number z, hence T is an $M(=3)$-hyponormal operator.

COROLLARY 3.13. *Let $T \in L(H)$ be in $\mathcal{L}$. If $(T-z)^n x=0$ for all real number z and some $n \geq 1$. Then $T^*x=\bar{z}x$.*

PROOF. It follows from Example 3.12 that T is a 3-hyponormal operator. From Lemma 2.1 (2) it is clear that $||(T-z)x||^{n+1} \leq 3^{\frac{n(n+1)}{2}} ||(T-z)^{n+1}x||$, and so $||(T-z)x||^{n+1}=0$ implies $Tx=zx$. Thus, in view of Lemma 2.1 (1), $Tx=zx$ implies $T^*x=\bar{z}x$.

4. An operator of M-power class (N)

We consider the following subset of M-hyponormal operators; T satisfies the addition property that for all z in the complex plane, all integers n and all $x \in H$, $||(T-z)^n x||^2 \leq M||(T-z)^{2n}x|| \, ||x||$. We call an operator with these properties an operators of M-power class (N). [8]. From a class of operators on H the operator T is said to be of class (N) if $x \in H$, $||x||=1$, $||Tx||^2 \leq ||T^2x||$. [9].

LEMMA 4.1. [6, Lemma 2] *Let T be of class (N). Then $||T^{n+1}x||^2 \geq ||T^n x||^2 ||T^2 x||$ for every unit vector $x \in H$ and $n \geq 1$.*

LEMMA 4.2. *If T is of M-power class (N), then the spectral radius $r(T)$ of T is not equal to $||T||$. But if T is of class (N), then $||T||=r(T)$.*

PROOF. We can assume without loss of generality that $||T|| =1$. It follows from [9, Lemma] that there exists a sequence $\{x_n\}$ such that $||x_n||=1$ and $\lim ||Tx_n||=1$. Since T is of M-power class (N), it is obvious that $1=\lim_{n\to\infty}||Tx_n||^2 \leq M \lim_{n\to\infty} ||T^2x_n||$ which yields $||T^2|| \geq \frac{1}{M}$. For all integer n we have

$||T^{2n}|| \geq \frac{1}{M2^n-1}$ by induction. Thus $r(T)=\lim_{n\to\infty}||T^n||^{\frac{1}{n}}=\lim_{n\to\infty}||T^{2m}||^{\frac{1}{2m}}$, $(n=2m)$, $\geq \lim_{m\to\infty}\left(\frac{1}{M^{2^m-1}}\right)^{\frac{1}{2m}}=1$. Since $||T||=1$, $\frac{1}{M}||T||=\frac{1}{M}\leq 1\leq r(T)$, that is, $\frac{1}{M}||T||\leq r(T)$. If T is of class (N), then the inequality $||Tx||^n\leq||T^nx||$ holds for $n=2$. Suppose that for the case $n=k$ the inequality $||Tx||^k\leq||T^kx||$ holds. If $n=k+1$, then it follows Lemma 4.1 that $||T^{k+1}x||^2\geq||T^kx||^2||T^2x||\geq||Tx||^{2k}||Tx||^2=||Tx||^{2(k+1)}$, and so the inequality $||T^{k+1}x||\geq||Tx||^{k+1}$ holds. Therefore, we have $r(T)=\lim ||T^n||^{\frac{1}{n}}=||T||$

Example 4.3. [8]. From Example 2.5 T is of M-power class (N) and $r(T)\neq||T||$.

Example 4.4. Let T be a hyponormal operator. Then T is of class (N) and $r(T)=||T||$.

Proof. Since $||Tx||^2=(Tx, Tx)=(T^*Tx,x)\leq||T^*Tx||\leq||T^2x||$, T is of class (N). It is clear, by Lemma 4.2, that $r(T)=||T||$.

Lemma 4.5. [8, Theorem 2.1] *If T is of M-power class (N) and $T^{-1}\in L(H)$, then T^{-1} is also of M-power class (N).*

Theorem 4.6. *If T is of M-power class (N) and $T^{-1}\in L(H)$, then $m_T=\{x:||T^{-n}x||\leq M||x||,\ n=2,3,\cdots,\ ||T^{-1}x||=M\}$ is invariant under T^{-1}*

Proof. If T is of M-power class (N), it follows from Lemma 4.5 that T^{-1} is of M-power class (N). Let $x\in m_T$ and $||x||=1$. Then we have $||T^{-n}x||^2\leq M||T^{-2n}x||$, and thus also $||T^{-n}(T^{-1}x)||^2\leq M||T^{-2(n+1)}x||$. It is clear, from the Definition of m_T that $||T^{-2n-1}||=\sup_{||x||=1}||T^{-2n-1}x||\leq M$. Thus we have $\frac{1}{M}||T^{-(n+1)}x||^2 \leq ||T^{-2(n+1)}x|| \leq ||T^{-2n-1}||\,||T^{-1}x|| \leq M$ $||T^{-1}x||$. Since $||T^{-1}x|| = M \geq 1$, it follows that $||T^{-n}(T^{-1}x)||^2 \leq M^2||T^{-1}x|| \leq M^2||T^{-1}x||^2$, hence $||T^{-n}(T^{-1}x)|| \leq M||T^{-1}x||$. Therefore, $T^{-1}x \in m_T$, m_T is invariant under T^{-1}.

REMARK 4.7. If T is of M-power class (N) and $T^{-1} \in L(H)$, then $||(T-z)^{*-1}x||^2 \leq M^3||(T-z)^{-2}x||$ for all $x \in H$, $||x||=1$, and for all z in resolvent set of T, $\rho(T)$.

PROOF. It follows from Lemma 4.5 that T^{-1} is of M-power class (N), and so we have $||(T-z)^{-1}x||^2 \leq M||(T-z)^{-2}x||$ for all $z \in \rho(T)$ and for $n=1$. Since T is M-hyponormal, it is clear, by the inequality; $(T-z)(T-z)^* \leq M^2(T-z)^*(T-z)$, that $(T-z)^{-1}(T-z)^{*-1} \leq M^2(T-z)^{*-1}(T-z)^{-1}$ holds, hence it follows that $||(T^*-\bar{z})^{-1}x|| \leq M||(T-z)^{-1}x||$ for all $z \in \rho(T)$. Thus, we have $||(T^*-\bar{z})^{-1}x||^2 \leq M^2||(T-z)^{-1}x||^2 \leq M^3||(T-z)^{-2}x||$ for all $z \in \rho(T)$.

References

[1] Stephen L. Campbell and Ralph Gellar, Spectral properties of linear operators for which T^*T and $T+T^*$ commute, Proc. A. M. S, 60, (1976), 197-202.

[2] R.G. Douglas, On majorization, factorization, and range inclusion of operators on a Hilbert space, Proc. A.M.S, 17, (1966), 413-415.

[3] H.R. Dowson, Spectral theory of linear operators, Academic Press, 1978.

[4] Peter A. Fillmore, Notes on operator theory, Van Nostrand Reinhold Com., 1970.

[5] Masatoshi Fujii and Yasuhiko Nakatsu, On subclasses of hyponormal operators, Proc. Japan Acad., 51, (1975), 243-246.

[6] Takayuki Furuta, On the class of paranormal operators, Proc. Japan Acad., 43, (1967), 594-598.

[7] Gilbert Helmberg, Introduction to spectral theory in Hilbert space, North-Holland Pub. Com., 1967.

[8] Vasile I. Istrătescu, Some results on M-hyponormal operators, Mathematics Seminer Notes, 6, (1978), 77-86.

[9] ——————, On some hyponormal operators, Paci. J. M., 22, (1967), 413-417.

[10] Che-Kao Fong and V. I. Istrătescu, Some characterizations of hermition operators and related classes of operators, I, Proc. A. M. S, 71, (1979), 107-112.

[11] Che-kao Fong, On M-hyponormal operators, Studia Mathematica, LXV, (1979), 1-5.

[12] R. L. Moore, D. D. Rogers and T. T. Trent, A note on intertwining M-hyponormal operators, Proc. A. M. S, 83, (1981), 514-516.

[13] Bhushan L. Wadhwa, M-hyponormal operator, Duke. Math, 41, (1974), 655-660.

Department of Mathematics
University of Ulsan.
Ulsan 690
Korea