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SOME REMARKS ON THE MULTIPLICATORS FOR H AND THE CONVOLUTION OPERATORS IN H'

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0, Introduction

L. Schwartz[7] determined the multiplicators and the convolutors for the tempered distributions. Hasumi[2] determined the multiplicator for $\mathfrak{H}$ and the convolutor for H'. Z. Zieleźny[9] studied convolution operators more precisely and concretely in H'. M. Morimoto[3] determined the convolutors for the space of Fourier ultrahyperfunctions.

In this paper, we considered the dual space $\mathcal{O}_c(\mathcal{S}', \mathcal{S}')$ (resp. $\mathcal{O}_c(\mathrm{H}', \mathrm{H}')$) of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$(resp. $\mathcal{O}_c'(\mathrm{H}', \mathrm{H}')$). We can write down the relations of various spaces:

$\mathcal{O}_M(\mathfrak{H}, \mathfrak{H})\subset\mathcal{O}_c(\mathcal{S}', \mathcal{S}')\subset\mathcal{O}_M(\mathcal{S}, \mathcal{S})\subset\mathcal{O}_c(\mathrm{H}', \mathrm{H}')\subset\mathcal{O}_M(\mathrm{H}, \mathrm{H})$.

We also examined the topology on $\mathcal{O}_c'(\mathrm{H}', \mathrm{H}')$ by some different way of [9]. We considered the representation of the space $\mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$.

We will examine the convolutors for the space of Fourier ultrahyperfunctions in the forthcoming paper.

1. The spaces $H(R^n)$ and $H'(R^n)$

we recall some definitions and properties on the spaces $H(R^n)$ and $H'(R^n)$ to clarify our problems in this paper.

$H(R^n)$ is the space of all C^∞-functions $\varphi(x)$ on R^n such that $\exp(k|x|)D^p\varphi(x)$ is bounded on R^n for any nonnegative integer k and multi-index p. A fundamental system of seminorms in $H(R^n)$ is defined by

$$\|\varphi\|_k=\sup\{\exp(k|x|)|D^p\varphi(x)|;0\leqq|p|\leqq k,x\in R^n\}$$

for $k=0,1,2,\cdots$.

The space $\mathcal{D}(R^n)$ of C^∞-functions with compact support is dense in the space $H(R^n)$. The space $H(R^n)$ is Fréchet nuclear and reflexive.

The dual space $H'(R^n)$ of $H(R^n)$ is the subspace of the space $\mathcal{D}'(R^n)$ of distributions on R^n whose elements are distributions with exponential growth([5]). The topology of $H'(R^n)$ is the strong dual topology; it makes $H'(R^n)$ into a complete, locally convex and Montel space.

For a function $\varphi\in H(R^n)$, its Fourier transform

$$\widehat{\varphi}(\zeta)=\int\cdots\int_{R}\exp(-i<x,\zeta>)\varphi(x)dx_1\cdots dx_n$$

is defined for all $\zeta\in C^n$. We denote by $\mathfrak{H}(C^n)$ the space of Fourier transforms of functions from $H(R^n)$. $\mathfrak{H}(C^n)$ consists of all entire functions rapidly decreasing in any tube, with compact base. In other words, an entire function ψ is in $\mathfrak{H}(C^n)$ if and only if, for any polynomial $P(z)$ of z and any compact set K of R^n, $|P(z)\psi(z)|$ is bounded for $z\in T(K)=R^n\times iK$.

A fundamental system of seminorms in $\mathfrak{H}(C^n)$ is defined by

$$p_k(\psi)=\sup_{z\in T_k}\{|z^k\psi(z)|\}$$

for $k=0,1,2,\cdots$, where $z^k=z_1{}^k\cdots z_n{}^k$ and $T_k=\{z\in C^n;z=x+iy,\ |y_j|\leqq k$ for $j=1,2,\cdots,n\}$.

The Fourier transformation $\mathcal{F}$ is a topological isomorphism of $H(R^n)$ onto $\mathfrak{H}(C^n)$. The inverse Fourier transformation is given by the following formula:

$$\overline{\mathcal{F}}\psi(x)=\frac{1}{(2\pi)^n}\int\cdots\int_{R^n}\exp(i<x,\xi>)\psi(\xi)d\xi_1\cdots d\xi_n.$$

The space $\mathfrak{H}(C^n)$ is Fréchet nuclear and therefore reflexive. If we define the Fourier transformation $\mathcal{F}$ on $\mathfrak{H}'(C^n)$ by the duality, the Fourier image of $\mathfrak{H}'(C^n)$ coincides with the space $H'(R^n)$.

A distribution $T\in\mathcal{D}'(R^n)$ is $H'(R^n)$ if and only if T can be represented in the form

$$T=D^p[\exp(k|x|)f(x)],$$

where $p\in N^n$, $k\in R$ and f is a bounded, continuous function on R^n. Or equivalently, $T\in H'(R^n)$ if and only if, each regularization $T*\mathcal{L}$, $\mathcal{L}\in\mathcal{D}(R^n)$, is a continuous function of exponential growth; in that case there is a $k\in N$ such that

$$(T*\mathcal{L})(x)=0(\exp(k|x|))$$

as $|x|\to\infty$, for all $\mathcal{L}\in\mathcal{D}(R^n)$ (see[9]).

2. $\mathcal{O}_M(\mathcal{S},\mathcal{S})$ and $\mathcal{O}_c(\mathcal{S}',\mathcal{S}')$

Let $\mathcal{H}'$ be a space of distributions in R^n, which may be the space $\mathcal{D}'(R^n)$ or one of its subspaces with a topology stronger than that induced in $\mathcal{H}'$ by $\mathcal{D}'(R^n)$. For instance, $\mathcal{H}'$ is $\mathcal{D}'(R^n)$ or $H'(R^n)$ or $\mathcal{S}'(R^n)$.

Furthermore, let $\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')$ be the space of convolution operators in $\mathcal{H}'$, i.e., the space of continuous linear mappings of $\mathcal{H}'$ into $\mathcal{H}'$ which are convolution operators on $\mathcal{E}'\subset\mathcal{H}'$.

We identify the space $\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')$ of convolution operators in $\mathcal{H}'$ with the space of distributions, which consists of all $S\in\mathcal{H}'$ such that the mapping $T\to S*T$ of $\mathcal{E}'$ into $\mathcal{H}'$ can be extended to a continuous linear mapping of $\mathcal{H}'$ into $\mathcal{H}'$. Therefore, $\mathcal{E}'\subset\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')\subset\mathcal{H}'$. With the topology in $\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')$ induced by the space $L_b(\mathcal{H}',\mathcal{H}')$, the injection $\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')\to\mathcal{H}'$ is continuous and the bilinear mapping $(S,T)\to S*T$ of $\mathcal{O}_c'(\mathcal{H}',\mathcal{H}')\times\mathcal{H}'$ into $\mathcal{H}'$ is separately continuous.

For $S_1, S_2 \in \mathcal{O}_c'(\mathcal{H}', \mathcal{H}')$, the convolution $S_1 * S_2$ is also in $\mathcal{O}_c'(\mathcal{H}', \mathcal{H}')$. Moreover, the bilinear mapping $(S_1, S_2) \to S_1 * S_2$ of $\mathcal{O}_c'(\mathcal{H}', \mathcal{H}') \times \mathcal{O}_c'(\mathcal{H}', \mathcal{H}')$ into $\mathcal{O}_c'(\mathcal{H}', \mathcal{H}')$ is separately continuous.

We now consider the case $\mathcal{H}' = \mathcal{S}'(R^n)$. $\mathcal{E}\mathcal{S}'$ is the set of all C^∞-functions $f \in \mathcal{S}'(R^n)$ such that, for any $S \in \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$, the convolution $S*f$ is a C^∞-function and $S \to S*f$ is a continuous mapping of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ into $\mathcal{E}$. $\mathcal{A}\mathcal{S}'$ is a subset of $\mathcal{E}\mathcal{S}'$. A function $f \in \mathcal{E}\mathcal{S}'$ is in $\mathcal{A}\mathcal{S}'$, if, for every $S \in \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$, the convolution $h = S*f$ can be continued analytically in the complex n-space C^n and the growth of the resulting entire function is restricted in the following way. In any horizontal band T_k in C^n around R^n of width k, $|h(z)| \leq |g(Re\ z)|$, where g is a function of $\mathcal{E}\mathcal{S}'$ depending on k and $Re\ z$ is the real part of z. For each $f \in \mathcal{E}\mathcal{S}'$ and $S \in \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$, clearly $f*S \in \mathcal{E}\mathcal{S}'$ *i.e.*, $\mathcal{E}\mathcal{S}'$ is a module over $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ under the convolution operation.

A function $f(x)$ defined on R^n is *slowly increasing*, if there is a constant k such that

$$f(x) = 0(|x|^k), \tag{2.1}$$

as $|x| \to \infty$; $f(x)$ is *rapidly decreasing*, if condition (2.1) holds for every negative k.

A distribution $S \in \mathcal{D}'(R^n)$ is rapidly decreasing, if and only if, for any integer k, $(1+||x||^2)^{k/2} S(x)$ is a bounded distribution, or equivalently for every $k \geqq 0$, S is a finite sum of derivatives of continuous functions, whose products with $|x|^k$ are bounded in R^n. $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ is the space of rapidly decreasing distributions.

One refers the elements of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ as the distributions rapidly decreasing at infinity.

Fourier transforms of distributions of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$from the space $\mathcal{O}_M(\mathcal{S}, \mathcal{S})$ of C^∞-functions, slowly increasing together with all their derivatives. A C —function f is in $\mathcal{O}_M(\mathcal{S}, \mathcal{S})$, if and only if, for any multiindex p, there exists a polynomial P such that $|D^p f| \leq |P|$ on R^n. The topology of $\mathcal{O}_M(\mathcal{S}, \mathcal{S})$ is such that the Fourier transform is a topological isomorphism of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ onto $\mathcal{O}_M(\mathcal{S}, \mathcal{S})$. Moreover, the convolution $S*T$ of $S \in \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ and $T \in \mathcal{S}'(R^n)$ is transformed into the product $\hat{S}\hat{T}$ i. e., $\mathcal{F}(S*T) = (\mathcal{F}S)(\mathcal{F}T)$. On the other hand, th Fourier transformation $\mathcal{F}$ gives a topological linear isomorphism:

$$\mathcal{O}_M(\mathcal{S}, \mathcal{S}) \xrightarrow{\mathcal{F}} \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$$

If $f \in \mathcal{O}_M(\mathcal{S}, \mathcal{S})$ and $T \in \mathcal{S}'(R^n)$, then $\mathcal{F}f \in \mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ and $\mathcal{F}T \in \mathcal{S}'(R^n)$ and we have $\mathcal{F}(fT) = (\mathcal{F}f)*(\mathcal{F}T)$.

The set $\mathcal{E}\mathcal{S}'$ coincides with the space $\mathcal{O}_c(\mathcal{S}', \mathcal{S}')$ of very slowly increasing C^∞-functions, which is the dual of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$. (See[8]).

Recall that a C^∞-function f is in $\mathcal{O}_c(\mathcal{S}', \mathcal{S}')$, if and only if its derivatives $D^r f$ have the same rate of increase as a power of $|x|$. In another words, there exists a constant k, such that

$$D^r f(x) = 0(|x|^k)$$

as $|x| \to \infty$, for all the derivatives.

The set $\mathcal{A}\mathcal{S}'$ consists of functions $f \in \mathcal{E}\mathcal{S}'$ extendable over C^n as entire functions, slowly increasing in any horizontal band T_k. More precisely, an entire function f is in $\mathcal{A}\mathcal{S}'$, if and only if, in every band T_k,

$$|f(z(| \leq M(1+|z|^\mu),$$

wher M and μ are constants depending on k

THEOREM 2.1. *$\mathcal{E}\mathcal{S}'=\mathcal{O}_c(\mathcal{S}',\mathcal{S}')\subset\mathcal{O}_M(\mathcal{S},\mathcal{S})$, i.e., the set $\mathcal{E}\mathcal{S}'$ coincides with $\mathcal{O}_c(\mathcal{S}',\mathcal{S}')$ and it is a subspace of $\mathcal{O}_M(\mathcal{S},\mathcal{S})$.*

$f\in\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ if and only if $f\in\mathcal{O}(C^n)$, for any $h>0$ there exists a multiindex p such that $|f(z)|(1+|z|^p)^{-1}$ is boundedon T_h.

THEOREM 2.2. *The set $\mathcal{A}\mathcal{S}'$ coincides with $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ and $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})\subset\mathcal{O}_c(\mathcal{S}',\mathcal{S}')\subset\mathcal{O}_M(\mathcal{S},\mathcal{S})$. Moreover $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ is a module over $\mathcal{O}_c'(\mathcal{S}',\mathcal{S}')$ under convolution.*

3. The space $\mathcal{O}_c'(H',H')$ of convolution operators in H'

For $S\in H'(R^n)$ and $T\in\mathcal{E}'$, the convolution $S*T$ is well defined as a distribution in $H'(R^n)$ and $T\to S*T$ is a continuous linear mapping from $\mathcal{E}'$ into $H'(R^n)$. We call S a convolution operator in $H'(R^n)$, if the latter mapping is continuously extendable to a mapping from $H'(R^n)$ into $H'(R^n)$.

We denote by $\mathcal{O}_c'(H',H')$ the linear space of all convolution operators in $H'(R^n)$. One refers the elements of $\mathcal{O}_c'(H',H')$ as the distributions very rapidly decreasing at infinity.

PROPOSITION 3.1. [9]. *A distribution S is in $\mathcal{O}_c'(H',H')$ if and only if it satisfies one of the equivalent conditions:*

(a) *For every $a\in R^n$, the product $exp<a,x>S(x)$ is a bounded distribution on R^n, i.e., $exp<a,x>S(x)\in\mathcal{B}'$.*

(b) *For every $k\in N$, S can be represented as a finite sum of derivatives of continuous functions F_p,*

(1) $S=\sum_{|p|\leq m} D^pF_p,$

where

(2) $|F_p(x)|\leq M_p\ exp(-k|x|)$; *$M_p$ are constants.*

(c) *For every* $k \in N$, *the set of distributions* $exp(k|h|)\tau_h S$, $h \in R^n$, *is bounded in* $\mathcal{D}'(R^n)$.

(d) *For every* $\mathcal{L} \in \mathcal{D}(R^n)$, *the regularization* $S*\mathcal{L}$ *is in* $H(R^n)$.

Condition (d) *can be replaced by the stronger condition*:

(d′) *For every* $\varphi \in H(R^n)$, *the convolution* $S*\varphi$ *is in* $H(R^n)$ *and the mapping* $\varphi \to S*\varphi$ *of* $H(R^n)$ *into* $H(R^n)$ *is continuous.*

PROPOSITION 3.2. $\mathcal{O}_c'(H', H') \times H'(R^n) \ni (T, U) \to T*U \in H'(R^n)$ *is separately continuous.*

We denote by $\mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$ *the space of all* C^∞*-functions extendable over* C^n *as entire functions slowly increasing in any horizontal band. This means that an entire function* $\mathcal{L}$ *is in* $\mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$ *if and only if for each* $h>0$ *there exists a* $p \in N^n$ *such that*

$$|\mathcal{L}(z)|(1+|z^p|)^{-1} \text{ is bounded on } T_h.$$

$\mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$ *is the space of multiplication operators in* $\mathfrak{H}'(C^n)$. *If* $\varphi \in \mathfrak{H}(C^n)$ *and* $\mathcal{L} \in \mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$, *then* $\varphi\mathcal{L} \in \mathfrak{H}(C^n)$ *and the mapping* $\varphi \to \varphi\mathcal{L}$ *of* $\mathfrak{H}(C^n)$ *into* $\mathfrak{H}(C^n)$ *is continuous.*

The product $\mathcal{L}T$ *of* $\mathcal{L} \in \mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$ *and* $T \in \mathfrak{H}'(C^n)$ *is defined by equation*

$$(\mathcal{L}T)\cdot\varphi = T\cdot(\mathcal{L}\varphi), \quad \varphi \in \mathfrak{H}(C^n).$$

PROPOSITION 3.3. *The Fourier transformation* $\mathcal{F}$ *gives an isomorphism*:

$$\mathcal{O}_M(\mathfrak{H}, \mathfrak{H}) \xrightarrow{\mathcal{F}} \mathcal{O}_c'(H', H').$$

If $f \in \mathcal{O}_M(\mathfrak{H}, \mathfrak{H})$ *and* $T \in \mathfrak{H}'(C^n)$, *then* $\mathcal{F}f = \hat{f} \in \mathcal{O}_c'(H', H')$ *and* $\mathcal{F}T = \widehat{T} \in H'(R^n)$ *and we have* $\mathcal{F}(fT) = f\widehat{T} = (\mathcal{F}f)*(\mathcal{F}T) = \hat{f}*\widehat{T}$.

We define the space $\mathcal{O}_M(\mathcal{S}', \mathcal{S}')$ (*resp.* $\mathcal{O}_M(H', H')$) *as the image of Fourier transformation of the space*

$\mathcal{O}_c(\mathcal{S}',\mathcal{S}')$ (*resp.* $\mathcal{O}_c(H',H')$).

Summing up the results, we write down the table of various spaces:

$$\begin{array}{ccccccccc} \mathcal{O}_M(\mathfrak{H},\mathfrak{H}) & \subset & \mathcal{O}_c(\mathcal{S}',\mathcal{S}') & \subset & \mathcal{O}_M(\mathcal{S},\mathcal{S}) & \subset & \mathcal{O}_c(H',H') & \subset & \mathcal{O}_M(H,H) \\ \downarrow\mathcal{F} & & \downarrow\mathcal{F} & & \downarrow\mathcal{F} & & \downarrow\mathcal{F} & & \downarrow\mathcal{F} \\ \mathcal{O}_c'(H',H') & \subset & \mathcal{O}_M(\mathcal{S}',\mathcal{S}') & \subset & \mathcal{O}_c'(\mathcal{S}',\mathcal{S}') & \subset & \mathcal{O}_M(H',H') & \subset & \mathcal{O}_c'(\mathfrak{H}',\mathfrak{H}') \end{array}$$

4. The topology on the space $\mathcal{O}_c'(H',H')$

Let $L_b(H',H')$ be the space of all continuous linear mappings from $H'(R^n)$ into $H'(R^n)$ endowed with the topology of uniform convergence on all bounded sets. Denote by $\mathcal{T}$ (resp. $\mathcal{T}'$) the topology induced in $\mathcal{O}_c'(H',H')$ by $L_b(H',H')$ (resp. $L_b(H,H)$). Then the topologies $\mathcal{T}$ and $\mathcal{T}'$ in $\mathcal{O}_c'(H',H')$ coincide ([9]).

Let B (resp. B') be a bounded set in $H'(R^n)$ (resp. $H(R^n)$) and let

$$\|\varphi\|_B=\sup_{T\in B}|\langle T,\varphi\rangle| \quad (\text{resp. } \|T\|_{B'}=\sup_{\varphi\in B'}|\langle T,\varphi\rangle|)$$

for $\varphi\in H(R^n)$ (resp. $T\in H'(R^n)$). Then

$$U=U(B',\varepsilon)=\{T\in H'(R^n):\|T\|_{B'}<\varepsilon\}$$

$$(\text{resp. } U'=U'(B,\varepsilon)=\{\varphi\in H(R^n):\|T\|_B<\varepsilon\}$$

is a 0-neighborhood in $H'(R^n)$ (resp. $H(R^n)$).

Let $M(B,U)=\{S\in\mathcal{O}_c'(H',H'):S*T\in U \text{ for all } T\in B\}$ and let

$$M'(B',U')=\{S\in\mathcal{O}_c'(H',H'):S*\varphi\in U' \text{ for all } \varphi\in B'\}.$$

THEOREM 4.1. *For every bounded symmetric set B' (resp. B) and for every O-neighborhood U' (resp. U) in $H(R^n)$ (resp. $H'(R^n)$), there exists a set M (resp. M') in the O-neighborhood base for the topology $\mathcal{T}$ (resp. $\mathcal{T}'$) such that $M*B'\subset U'$ (resp. $M'*B\subset U$).*

PROOF. Let $M'(B',U')=\{S\in\mathcal{O}_c'(H',H'):S*\varphi\in U'$ for all $\varphi\in B'\}$, and let A be a bounded symmetric set in $H'(R^n)$.

We may assume that $U'=\{\varphi\in H(R^n):\|\varphi\|_A<\varepsilon\}$, $\varepsilon>0$. We choose a O-neighborhood $U^{\varepsilon}_{B'}$ in $H'(R^n)$: $U^{\varepsilon}_{B'}=\{T\in H'(R^n):\|T\|_{B'}<\varepsilon\}$.

Then, the set $M=\{S\in\mathcal{O}_c'(H',H')$: $S*T\in U^{\varepsilon}_{B'}$ for all $T\in A\}$ is a set in the O-neighborhood base for the toplogy $\mathcal{T}$. Moreover, if $S\in M$, then

$$\|S*T\|_{B'}<\varepsilon \text{ for all } T\in A$$

i.e., $|<S*T,\varphi>|<\varepsilon$ for all $\varphi\in B'$ and $T\in A$.
Hence, by the symmetry of B' and A,

$$|<T,S*\varphi>|<\varepsilon \text{ for all } \varphi\in B' \text{ and } T\in A.$$

Consequently, $S*\varphi\in U'$ for all $\varphi\in B'$ i.e., $M*B'\subset U'$.

COROLLARY 4.2. *By the above proof,* $\mathcal{T}'\subset\mathcal{T}$. *By the above latter statement,* $\mathcal{T}\subset\mathcal{T}'$. *Hence* $\mathcal{T}$ *coincides with* $\mathcal{T}'$.

REMARK 4.2 We denote by $L_s(H,H)$ the space $L(H,H)$ under the topology of simple convergence. The topology $\mathcal{T}'$ in $\mathcal{O}_c'(H',H')$ coincides with the topology induced in $\mathcal{O}_c'(H',H')$ by the space $L_s(H,H)$.

The topology $\mathcal{T}''$ in $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ coincides with the topology induced in $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ by the space $L_s(\mathfrak{H},\mathfrak{H})$.

For each $j,k\in N$, we denote by

$$\tilde{E}_{k,j}=\{\chi\in\mathcal{O}(T_k):p_{k,j}(\chi)=\sup_{\zeta\in T_k}(1+|\zeta|)^j|\chi(\zeta)|<\infty\}$$

(resp. $E_{k,j}=\{\psi\in\mathcal{O}(T_k):\|\psi\|_{k,j}=\sup_{\zeta\in T_k}(1+|\zeta|)^{-j}|\psi(\zeta)|<\infty\}$).

Then $\tilde{E}_{k,j}$(resp. $E_{k,j}$) is a Banach space with the norm $p_{k,j}$ (resp. $\|\cdot\|_{k,j}$)

For fixed k, the spaces $\tilde{E}_{k,j}$(resp. $E_{k,j}$), $j=1,2,\cdots$, form a decreasing (resp. increasing) sequence:

$$\tilde{E}_{k,1}\supset\tilde{E}_{k,2}\supset\cdots\supset\tilde{E}_{k,j}\supset\cdots(\text{resp. } E_{k,1}\subset E_{k,2}\subset\cdots\subset E_{k,j}\subset\cdots)$$

and the topology induced in $\tilde{E}_{k,j+1}$(resp. $E_{k,j}$) by $\tilde{E}_{k,j}$(resp.

$E_{k,j+1}$) is coarser than the topology of $\tilde{E}_{k,j+1}$(resp. $E_{k,j}$). We denote by

$$\tilde{E}_k=\lim_{j\to\infty} \text{pioj } \tilde{E}_{k,j}(\text{resp. } E_k=\lim_{j\to\infty} \text{ind } E_{k,j})$$

For $\chi\in\mathcal{O}(T_k)$(resp. $\psi\in\mathcal{O}(T_k)$), $\chi\in\tilde{E}_k$(resp. $\psi\in E_k$) if and only if for every (resp. some) $j\in N$, $p_{k,j}(\chi)<\infty$(resp. $\|\cdot\|_{k,j}<\infty$).

We note that $\tilde{E}_k$ is in duality with the space E_k and the canonical bilinear form of the duality is

$$(\chi,\psi)\to\langle\chi,\psi\rangle=\sup_{\zeta\in T_k}|\chi(\zeta)\psi(\zeta)|,\ \chi\in\tilde{E}_k,\ \psi\in E_k.$$

PROPOSITION 4.4. *We have the following: as sets,* $E=\mathcal{O}_M(\mathfrak{H},\mathfrak{H})=\bigcap_k E_k=\bigcup_j E_{\infty,j}$, *where* $E_{\infty,j}=\bigcap_k E_{k,j}$, $\tilde{E}=\bigcap_k \tilde{E}_k=\bigcap_j \tilde{E}_{\infty,j}$, *where* $\tilde{E}_{\infty,j}=\bigcap_k \tilde{E}_{k,j}$.

PROPOSITION 4.5[9]. $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})=\mathit{lim\ proj}_{k\to\infty}\ E_k$

Since $\mathcal{O}_c'(H',H')$ is aclosed subspace of the complete nuclear space $L_b(H,H)$, $\mathcal{O}_c'(H',H')$ is a complete nuclear space. Since $\mathcal{O}_c'(H',H')$ is isomorphic to $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ by the Fourier transformation, and since $\mathcal{O}_M(\mathfrak{H},\mathfrak{H})$ is bornologic, $\mathcal{O}_c'(H',H')$ is a bornologic, Montel space (see[9]).

5. $\mathcal{O}_c(H',H')$ and $\mathcal{O}_M(H,H)$

$\mathcal{O}_c'(\mathcal{S}',\mathcal{S}')=\{T\in\mathcal{D}'(R^n)$; for every integer k,

$(1+\|x\|^2)^{k/2}T(x)$ is a bounded distribution$\}$

is the space of convolution operators in $\mathcal{S}'(R^n)$.

One refers the elements of the space $\mathcal{O}_c'(\mathcal{S}',\mathcal{S}')$ as distribution rapidly decreasing at infinity.

$\mathcal{O}_c'(H',H')$ is a linear subspace of the space $\mathcal{O}_c'(\mathcal{S}',\mathcal{S}')$ and the imbedding $\mathcal{O}_c'(H',H')\to\mathcal{O}_c'(\mathcal{S}',\mathcal{S}')$ is continuous.

A C^∞-function f is in the dual $\mathcal{O}_c(\mathcal{S}', \mathcal{S}')$ of $\mathcal{O}_c'(\mathcal{S}', \mathcal{S}')$ if and only if there exists a constant μ such that

$$D^r f(x) = O(|x|^\mu)$$

as $|x| \to \infty$, for all the derivatives.

PROPOSITION 5. 1. [9]

$\mathcal{O}_c(H', H') = \{f \in C^\infty(R^n)$: $D^p f(x) = O(exp(k|x|))$ as $|x| \to \infty$ for all $p \in N^n$ and some $k \in N$(independent of p)$\}$ is the dual space of $\mathcal{O}_c'(H', H')$.

THEOREM 5. 2. *The space $\mathcal{O}_M(H, H)$ of multiplicators for H is given as follows:*

$\mathcal{O}_M(H, H) = \{f \in C^\infty(R^n)$: for every $p \in N^n$, there exists integer k, $C \geqq 0$ such that $|D^p f(x)| \leqq C exp(k|x|)$, $x \in R^n\}$

The space $\mathcal{O}_c(H', H')$ is a (linear) subspace of $\mathcal{O}_M(H, H)$.

PROPOSITION 5. 3. [9]. *The space $\mathcal{O}_c(H', H')$ endowed with the strong topology is a complete nuclear Montel space.*

References

[1] José Barros-Neto: An Introduction to the Theory of Distributions Marcel Dekker, Inc. New York 1973.

[2] Hasumi, M.: Note on the n-dimensional tempered ultra-distributions Tôhoku Math. J., 13 (1961), 94-104.

[3] Morimoto, M.: Convolutors for Ultrahyperfunctions, Lecture Notes in Physics No. 39 Springer.

[4] ————,: Theory of tempered ultrahyperfunctions, I. Proc. Japan Acad. Vol. 51, No. 2 (1975).

[5] Park, Y.S. and M. Morimoto: Fourier ultra-hyperfunctions in the Euclidean n-space, J. Fac. Sci. Univ. Tokyo Sec. IA, 20 (1973), 121-127.

[6] Park, Y.S.: Fourier Ultra-Hyperfunctions Valued in a Fréchet Space Tokyo J. of Math. Vol. 5, No. 1, June 1982, 143-155.

[7] Schwartz, L.: Théorie des Distributions, Hermann, Paris, 1967.

[8] Zieleźny, z. : Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions(I), Studia Math. 28 (1967) p. 317-332.

[9] ———,: On the space of convolution operators in $\mathscr{K}'_1$, Studia Math. 31 (1968), p. 111-124.

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