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SOME PROPERTIES OF BITOPOLOGICAL SPACES

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1. Introduction

A bitopological space (X, P, Q) is a set X together with two topologies P and Q on X. In this paper we study some properties of bitopological spaces.

In J. C. Kelly[2], Theorem 2.7. is the generalization of Urysohn's Lemma. In this paper we prove the sufficiency condition also holds.

In section 3, when we define a bitopological space (X, P, Q) to be pairwise paracompact if (X, P, Q) is pairwise Hausdorff and each P-open covering of X has a Q-open nbd-finite refinement, and each Q-open covering of X has a P-open nbd-finite refinement, we prove that every pairwise paracompact space is pairwise normal.

2. Pairwise Hausdorff, pairwise regular, pairwise normal bitopological spaces

The following definition extends to a bitopological space (X, P, Q) the notions of separation properties of a topological space (X, F).

DEFINITION 2.1. [2]. *In a space (X, P, Q), P is said to be regular with respect to Q if, for each point x in X, there is a P-neighbourhood base of Q-closed sets.*

THEOREM 2.2. *In a space (X, P, Q), P is regular with respect to Q if and only if, for each point x in X and* *each P-closed set P such that* $x \notin P$, *there are a P-open set U and a Q-open set V such that* $x \in U$, $P \subseteq V$, *and* $U \cap V = \phi$.

PROOF. Let x be an arbitrary point in X, and P is a P-closed set such that $x \notin P$. Then $x \in P^c$ and P^c is a P-open set. By hypothesis, there exists a P-neighbourhood of Q closed set U of x such that $U \subseteq P^c$. Then, $x \in U$, $P \subseteq U^c$, U is a P-open set, U^c is a Q-open set, and $U \cap U^c = \phi$, i.e. there exist a P-open set U and a Q-open set U^c such that $x \in U$, $P \subseteq U^c$, and $U \cap U^c = \phi$. Conversely, let x be an arbitrary point in X and $N(x)$ be a P-open neighbourhood base of x. Then for each element W of $N(x)$, W^c is P-closed and $x \notin W^c$

By hypothesis, there exist a P-open set U and a Q-open set V such that $x \in U$, $W^c \subseteq V$ and $U \cap V = \phi$. Then there are a P-open set U and Q-closed set V^c such that $x \in U \subset V^c \subset W$.

Put $W(x) = \{V^c\}$, then $W(x)$ is a P-neighbourhood base of Q-closed sets.

(X, P, Q) is or P and Q are, pairwise regular if P is regular with respect to Q and vice versa.

DEFINITION 2.3. [2]. *A space* (X, P, Q) *is said to be pairwise Hausdorff if for each two distinct points* x *and* y, *there are a P-neihbourhood U of x and a Q-neighbourhood V of y such* $U \cap V = \phi$.

THEROEM 2.4. [5]. *If a bitopological space* (X, P, Q) *is pairwise Hausdorff then sets which are compact with respect to one are closed with respect to the other.*

DEFINTION 2.5. [2]. *A space* (X, P, Q) *is said to be pairwise normal if, given a P-closed set A and a Q-closed set B* *with* $A \cap B=\phi$, *there exist a* Q*-open set* U *and a* P*-open set* V *such that* $A \subseteqq U$, $B \subseteqq V$, *and* $U \cap V=\phi$.

THEOREM 2.6. *A space*(X, P, Q) *is pairwise normal if and only if, given a* Q*-closed set* C *and a* P*-open set* D *such that* $C \subseteqq D$, *there are a* P*-open set* G *and a* Q*-closed set* F *such that* $C \subseteqq G \subseteqq F \subseteqq D$.

PROOF. Let C be a Q-closed set and D be a P-open set such that $C \subseteqq D$. Then D^c is a P-closed set and $C \cap D^c=\phi$. By hypothesis there exists a Q-open set U and a P-open set V such that $D^c \subseteqq U$, $C \subseteqq V$, and $U \cap V=\phi$.

Then $U^c \subseteqq D$, $C \subseteqq V$, $V \subseteqq U^c$, and U^c is a Q-closed set. Put $F=U^c$, and $G=V$, then there exist a P-open set G and a Q-closed set F such that $C \subseteqq G \subseteqq F \subseteqq D$.

Conversely, let A be a P-closed set and B be a Q-closed set such that $A \cap B=\phi$. Then A^c is a P-open set and B is a Q-closed set such that $B \subseteqq A^c$.

By hypothesis there exist a P open set G and Q-closed set F such that $B \subseteqq G \subseteqq F \subseteqq A^c$. Then $B \subseteqq G$, $A \subseteqq F^c$, F^c is a Q-open set and $G \cap F^c=\phi$. Thus there exist a P-open set G and a Q-open set F^c such that $B \subseteqq G$, $A \subseteqq F^c$, and $G \cap F^c=\phi$.

In J. C. Kelly [2], Theorem 2.7. is the generalization of Urysohn's Lemma. In this paper we prove that the sufficient condition also holds.

THEOREM 2.7. *A space* (X, P, Q) *is pairwise normal, if and only if, given a* Q*-closed set* F *and a* P*-closed set* H *with* $F \cap H=\phi$, *there exists a real-valued function* g *on* X *such that,*

[1] $g(x)=0$ $(x \in F)$, $g(x)=1$ $(x \in H)$, *and* $0 \leqq g \leqq 1$.

[2] g *is* P*-upper semi-continuous and* Q*-lower semi-con-*

tinuous.

PROOF. Necessity. [2, theorem 2.7]

Sufficiency. Let A and B be subsets of X such that A is a Q-closed set, B is a P-closed set, and $A\cap B=\phi$. By hypothesis there exists a real valued function g on X such that,

[1] $g(x)=0$ $(x\in A)$, $g(x)=1$ $(x\in H)$, and

[2] g is P-upper semi-continuous and Q-lower semi-continuous.

Put $U=\{x|g(x)<\frac{1}{2}\}$, $V=\{x|g(x)>\frac{1}{2}\}$. Then $A\subset U$, $B\subset V$, $U\cap V=\phi$, and U is a P-open set V is a Q-open set. Hence(X, P, Q) is pairwise normal.

Another necessary and sufficient conditions that (X, P, Q) is pairwise normal is in E. P. Lane [4].

THEOREM 2.8. *In order for (X, P, Q) to be pairwise normal, it is necessary and sufficient that for every pair of functions f and g defined on X such that f is P-lower semicontinuous and g is Q-upper semi-continuos, and $g\leqq f$, there exists a P-lower semi-continuous and Q-upper semicontinuous function h on X such that $g\leqq h\leqq f$.*

PROOF. Necessity. [4, theorem 2.5]

Sufficiency. Let A and B be subsets of X such that A is a P-closed set, B is a Q-closed set, $A\cap B=\phi$. Define real functions $\chi_{X-A}:X\to R, X_B:X\to R$ by $\chi_{X-A}(x)=\begin{cases}1 & x\in X-A,\\ 0 & x\in A\end{cases}$ $\chi_B(x)=\begin{cases}1 & x\in B\\ 0 & x\in X-B.\end{cases}$ Then χ_{X-A} is P-lower semi-continuous, χ_B is Q-upper semi-continuous and $\chi_B\leqq\chi_{X-A}$. By hypothesis there exists a P-lower semi-continuous and Q-upper semicontinuous function h on X such that $\chi_B\leqq h\leqq\chi_{X-A}$. Put $U=\{x:h(x)>\frac{1}{2}\}$, $V=\{x:h(x)<\frac{1}{2}\}$. Then $B\subset U$, $A\subset V$, $U\cap V$

$=\phi$, and U is a P-open set V is a Q-open set.

DEFINITION 2.9. *Suppose that P and Q are topologies on a set X. We say that P is completely regular with respect to Q in case every P-closed subset F of X and each point x in $X \setminus F$, there is a P-lower semicontinuous and Q-upper semicontinouus function f on X such that $f=0$ on F, $f(x)=1$, and $0 \leqq f \leqq 1$. The space (X, P, Q) is pairwise completely regular if P is P is completely regular with respect to Q and Q is completely regular with respect to P.*

DEFINITION 2.10. [4]. *A subset A of X is SC-embedded (resp. SC-embedded) in X if every real-valued (resp. bounded real valued) P-lower semicontinuous and Q-upper semicontinuous function on A can be extended to a P-lower semicontinuous and Q-upper semicontinuous function on X.*

THEOREM 2.11. [4]. *Every P-closed and Q-closed subset of a pairwise normal space (X, P, Q) is SC^*-embedded.*

DFFINITION 3.14. [4]. *Let (X, P, Q) be a bitopological space. If f is a real-valued function on X that is P-lower semicontinuous and Q-upper semicontinuous, then $\{x \in X | f(x) \leqq 0\}$ is a P-zero-set with respect to Q, and $\{x \in X | 0 \leqq f(x)\}$ is a Q-zero-set with respect to P.*

The terminology will be abbreviated as follows; a P-zero-set with respect to Q will be called a P-zero-set, and a Q-zero-set with respect to P will be called a Q-zero-set.

Let f be a P-upper semicontinuous and Q-lower semicontinuous function on X. Then, for every real number r, $\{x \in X | r \leqq f(x)\}$ is a P-zero-set and $\{x \in X | f(x) \leqq r\}$ is a Q-zero-set. Also, because $\{x \in X | g(x) \leqq 0\} = \{x \in X | (gV_0)(x)=0\}$, any P-zero-set is of the from $\{x \in X | h(x)=0\}$, where h is P-lower semicontinuous and Q-upper semicontinuous and $h \geqq$

0. Similarily, any Q-zero-set is of the from $\{x\in X | h(x)= 0\}$, where h is P-upper semicontinuous and Q-lower semicontinuous and $h\geqq 0$.

THEOREM 2.12. *The space (X, P, Q) is pairwise completely regular if and only if the P-zero-sets from a base for the P-closed sets and the Q-zero-sets from a base for the Q-closed sets.*

Before proving Theorem 2.12, we need the following Lemma.

LEMMA 2.13. *For every P-zero-set F and each point x in $X\setminus F$, there exists a P-lower semicontinuous and Q-upper semicontinuous function f on X such that $f(F)=0$, $f(x)=1$, and $0\leqq f(x)\leqq 1$ $(x\in X)$*

PROOF. Let F be a P-zero-set. Then $F=\{x\in X | h(x)=0\}$ where h is P-lower semicontinuous and Q-upper semicontinuous, and $h\geqq 0$. Let $x_0\in X\setminus F$, then by difinition of F, $h(x)>0$. Put $f(x)=\min\left\{\frac{h(x)}{h(x_0)}, 1\right\}$, then $f(F)=0$, $f(x_0)= 1$, and $0\leqq f\leqq 1$. Furthermore, f is P-lower semicontinuous and Q-upper semicontinuous. Since h is P-lower semicontinuous and Q-upper semicontinuous, and $h(x_0)>0$, $\frac{h(x)}{h(x_0)}$ is P-lower semicontinuous Q-upper semicontinuous. So $f(x) = \min\left\{\frac{h(x)}{h(x_0)}, 1\right\}$ is P-lower semicontinuous and Q-upper semicontinuous.

PROOF of THEOREM 2.12. Let F be a P-closed set and x be an arbitrary point in $X\setminus F$. By hypotesis, $F=\bigcap F_L$ where F_L is P-zero-set for each L. Since $F=\bigcap F_L$ and $x\notin F$, there exists a P-zero-set F_{L_0} such that $F\subset F_{L_0}$ and $x\notin F_{L_0}$. Then by Lemma 3.16, there exists a P-lower semicontinuous and

Q-upper semicontinuous function f on X such that $f(F_{L_0})=0$, $f(x)=1$, and $0\leqq f\leqq 1$. Then since $F\subset F_{L_0}$, $f(F)=0$. Thus F is P-completely regular with respect to Q. Similarily Q *is completely regular with respect to P.*

Conversely, let F be a P-closed set. Then since (X, P, Q) is pairwise completely regular, for each point x_0 in X-F_0 there is a P-lower semicontinuous and Q-upper semicontinuous *function* f_{x_0} *on* X such that $f_{x_0}(F)=0$, $f_{x_0}(x)=1$, and $0\leqq f_{x_0}\leqq 1$. Put $F_{x_0}=\{x\in X\,|\,f_{x_0}(x)\leqq 0\}$, then F_{x_0} is a Q-zero set and $F=\bigcap_{x_0\in X\setminus F} F_{x_0}$. Thus P-zero sets form a base for the P-closed sets. Similary Q-zero sets form a base for the Q-closed sets.

THEOREM 2.14. [4]. *If X is pairwise normal and if a subset A of X is a P-zero-set and a Q-zero-set, then A is SC-embedded in X*

3. Pairwise paracompact spaces

DEFINITION 3.1. *Let* $\{A_\alpha|\alpha\in A\}$ and $\{B_\beta|\beta\in B\}$ *be two coverings of a space X. $\{A_\alpha\}$ is said to refine (or be a refinement of) $\{B_\beta\}$ if for each A_α there is some B_β with $A_\alpha\subset B_\beta$.*

DEFINITION 3.2. *A refinement $\{A_\alpha|\alpha\in A\}$ of $\{B_\beta|\beta\in B\}$ is called precise if $A=B$ and $A_\alpha\subset B_\beta$ for each α.*

LEMMA 3.3. *If the space (X, P, Q) is pairwise Hausdorff. Then for a fixed point p in X, and for each point $q\neq p$ in X, there is a P-open neighbourhood $U_{(p)}$ such that $q\notin Q\text{-}clU_{(p)}$. Similarily there is a Q-open neighbourhood $V_{(p)}$ such that $q\notin P\text{-}clV_{(p)}$*

PROOF. Let q be an arbitrary point in X such that $p\neq q$. Since (X, P, Q) is pairwise Hausdorff there exist a P-open neighbourhooh $U_{(p)}$ and a Q-open neighbourhood $V(q)$ such that $U_{(p)} \cap V_{(q)} = \phi$. Then $U_{(p)} \subset V^c_{(q)}$, $q \notin V^c_{(q)}$, and $V^c_{(q)}$ is Q-closed. Thus $Q\text{-}cl\,U_{(p)} \subset V^c_{(q)}$. So $q \notin Q\text{-}cl\,U_{(p)}$.

LEMMA 3.4. *If the P-covering* $\{A_\alpha | \alpha \in A\}$ *of* X *has a Q-neighbourhood-finite refinement* $\{B_\beta | \beta \in B\}$, *then it also has a precise Q-neighbourhood-finite refinement* $\{C_\alpha | \alpha \in A\}$. *Furthermore, if each* B_β *is a Q-open set, then each* C_α *can be chosen to be an Q-open set also.*

PROOF. Define a map $\varphi: B \to A$ by assigning to each $\beta \in B$ some $\alpha \in A$ such that $B_\beta \subset A_\alpha$. For each $\alpha \in A$ let $C_\alpha = \bigcup \{B_\beta | \varphi(\beta) = \alpha\}$, some C_α may be empty. Clearly, $C_\alpha \subset A_\alpha$ for each α, and also $\{C_\alpha | \alpha \in A\}$ is a covering because each B_β appears somewhere; C_α is evidently Q-open whenever each B_β is Q-open. If $\{B_\beta\}$ is Q-neighbourhood-finite, then each point x in X has a Q-neighbourhood V such that $V \cap B_\beta \neq \phi$ for at most finitely many indices β. And the number of $\{C_\alpha\}$ such that $V \cap C \neq \phi$ is less then $\{B_\beta\}$. Thus $V \cap C_\alpha \neq \phi$ for at most finitely many indices α, $\{C_\alpha\}$ is therefore Q-neighbourhood-finite refinement.

LEMMA 3.5. [1] *Let* $\{A_\alpha | \alpha \in A\}$ *be a neighbourhood-finite family in* X. *Then*;

[1] $\{cl A_\alpha | \alpha \in A\}$ *is also neighbourhood-finite.*

[2] *for each* $B \subset A$, $\bigcup \{A_\beta | \beta \in B\}$ *is closed in* X.

DEFINITION 3.6. *A pairwise Hausdorff space* (X, P, Q) *is pairwise paracompact if each P-open covering of X has a Q-open neighbourhood finite refinement and each Q-open covering of X has a P-open neighbourhood-finite refinement.*

THEOREM 3.7. *Every pairwise paracompact space is pairwise normal.*

PROOF. We first show that the pairwise paracompact space is pairwise regular. Let A be a P-closed set in X and y be a given point in X such that $y \notin A$. Since (X, P, Q) is pairwise Hausdorff. We find by Lemma 3.3, that each $a \in A$ has a P-open neighbourhood U_a with $y \notin Q\text{-}clU_a$. Since $\{U_a| a \in A\} \cup \{A^c\}$ is a P-open covering of X, we use pairwise paracompactness and Lemma 3.4, to get a precise Q-neighbourhood-finite open refinement $\{V_a|a \in A\} \cup G$. Then $M = \cup \{V_a|a \in A\}$ is Q-open and contains A. Furthermore, because $\{V_a\}$ *is* Q-neighbourhood-finite, Lemma 3.5. shows that $Q\text{-}clM = \cup \{Q\text{-}clV_a|a \in A\}$ and since $y \notin Q\text{-}clU_a \supset Q\text{-}clV_a$ for each $a \in A$, we find $y \notin Q\text{-}clM$. Since (X, P, Q) is pairwise Hausdorff, and $y \notin Q\text{-}clM$, we find by Lemma 3.3. that each $b \in Q\text{-}clM$ has a Q-neighbourhood W_b with $y \notin P\text{-}clW_b$, since $\{W_b|b \in Q\text{-}clM\} \cup \{[Q\text{-}clM]^c\}$ is a Q-open covering of X, we use pairwise paracompactness and Lemma 3.4. to get a precise P-neighbourhood-finite open refinement $\{T_b|b \in Q\text{-}clM\} \cup H$. Then $N = \cup \{T_b|b \in Q\text{-}clM\}$ is P-open contains $Q\text{-}clM$. Furthermore, because $\{T_b\}$ is P-neighbourhood finite Lemma 3.5. shows that $P\text{-}clN = \cup \{P\text{-}clT_b|b \in Q\text{-}clM\}$, and since $y \notin P\text{-}clW_c \supset P\text{-}clT_b$ for each $b \in Q\text{-}clM$, we find $y \notin P\text{-}clN$. Then $y \in [P\text{-}clN]^c$, $[P\text{-}clN]^c \cap M = \phi$. i.e. there exist a P-open set $[P\text{-}clN]^c$, Q-open set M, $y \in [P\text{-}clN]^c$, $A \subseteq M$ such that $[P\text{-}clN]^c \cap M = \phi$. So P is regular with respect to Q, similarily Q is regular with respect to P. Thus (X, P, Q) is pairwise regular. We now prove that X is pairwise normal. Let A be a P-closed set, B be a Q-closed set with $A \cap B = \phi$. Then for each $b \in B$, $b \notin A$. Since (X, P, Q) is pairwise Hausdorff, we proved above that for each $b \in B$ there is a Q-open set M_b such that $A \subseteq M_b$ and $b \notin Q\text{-}clM_b$.

Then $A\subseteqq\bigcap_{b\in B} Mb$, $[\bigcap_{b\in B} Q\text{-}clMb]\cap B=\phi$. Put $M=\bigcap_{b\in B} M_b$, then $A\subseteqq M\subseteqq Q\text{-}clM$ and since $Q\text{-}clM=Q\text{-}cl[\bigcap_{b\in B} Mb]\subseteqq\bigcap_{b\in B} Q\text{-}clMb$, $Q\text{-}clM\cap B=\phi$. Since (X, P, Q) is pairwise regular, for each $a\in Q\text{-}clM$, there is a Q-open ses U_a and P-open set V_a such that $a\in U_a$, $B\subset V_a$, and $U_a\cap V_a=\phi$. Therefore for each $a\in Q\text{-}clM$, there is a Q-open set U_a such that $a\in U_a$ and $P\text{-}clU_a\cap B=\phi$. Since $\{U_a|a\in Q\text{-}clM\}\cup\{[Q\text{-}clM]^c\}$ is an Q-open covering of X, we use pairwise paracompactness and Lemma 3.4 to get a precise P-neighbourhood-finite open refinement $\{V_a|a\in Q\text{-}clM\}\cup G$. Put $N=\cup\{V_a|a\in Q\text{-}clM\}$, then N is P-open and $Q\text{-}clM\subseteqq N$. Furthermore, because $\{V_a\}$ is P-neighbourhood-finite, Lemma 3.5 shows that $P\text{-}clN=\cup\{P\text{-}clV_a|a\in Q\text{-}clM\}$. Since $P\text{-}clU_a\supset P\text{-}clV_a$ for each $a\in Q\text{-}clM$, and $B\cap P\text{-}clU_a=\phi$, we find $B\cap P\text{-}clN=\phi$. So we find a Q-closed set $Q\text{-}clM$, P-closed set N^c such that $A\subseteqq Q\text{-}clM$, $B\subset N^c$, and $Q\text{-}clM\cap N^c=\phi$. Since $Q\text{-}clM$ is Q-closed, N^c is P-closed, and $Q\text{-}clM\cap N^c=\phi$, as we proved above there is a set H uch that $N^c\subseteqq H\subseteqq Q\text{-}clH$ and $Q\text{-}clH\cap Q\text{-}clM=\phi$. Then $A\subseteqq Q\text{-}clM\subseteqq[Q\text{-}clH]^c$, $B\subseteqq[P\text{-}clN]^c$ Since $[P\text{-}clN]^c\subset N^c\subseteqq H\subseteqq Q\text{-}clH$, $[Q\text{-}clH]^c\cap[P\text{-}clN]^c=\phi$. In other word, there exist P-open set $[P\text{-}clN]^c$ and Q-open set $[Q\text{-}clH]^c$ such that $A\subseteqq[Q\text{-}clH]^c$, $B\subseteqq[P\text{-}clN]^c$ and $[Q\text{-}clH]^c\cap[P\text{-}clN]^c=\phi$.

EXAMPLE 3.8. In a real line R, let P be the usual topology and Q be the topology generated by the open-closed interval $(a, b]$ $(a, b\in R, a<b)$. Then (R, P, Q) is a paiwise paracom pact space.

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