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ON THE EXISTENCE RESULT FOR AN ABSTRACT NONLINEAR DIFFERENTIAL EQUATION

JONG SOO JUNG

1. Introduction

Let X be a real Banach space with norm $\|\ \|$ and $U\subset X$ be an open set.

The purpose of this paper is to study the local existence of the integral solution in the sense of Benilan [2], for the initial value problem

$$(E)\qquad \frac{du(t)}{dt}+Au(t)\ni F(u)(t),\ 0\leqq t\leqq T,$$

$$u(0)=u_0,$$

where $A\subset X\times X$ is a m-accretive set, F is a mapping from $C(0,T,U)$ into $C(0,T:X)$ and $u_0\in\overline{D(A)}\cap U$.

Under different assumptions than ours, this problem has been studied in [1, 6, 7, 9].

Section 2 is a preliminary part. In section 3, we establish the existence of integral solution of problem (E) in the case which $(I+\lambda A)^{-1}$ is compact for all $\lambda>0$ and F satisfies an appropriate bounded condition. Section 4 is devoted to the contiuation of solution and an example.

2. Preliminaries

Throughout this paper, X is a real Banach space with norm $\|\ \|$ and X^* is its dual space with the corresponding norm $\|\ \|_*$. Let $J:X\to 2^{X^*}$ be the duality mapping, i.e.:

(2.1) $J(x)=\{x^*\in X^*: <x, x^*> =||x||^2=||x^*||_*^2\}$

for each $x\in X$. For each $(x,y)\in X\times X$, define

(2.2) $\langle y, x\rangle_s=\sup\{\langle y, x^*\rangle : x^*\in J(x)\}$.

We recall that the mapping $\langle , \rangle_s : X\times X\rightarrow R$ is upper semicontinuous. If $A\subset X\times X$ and $x\in X$, we denote by $Ax=\{y\in X: (x,y)\in A\}$, $D(A)=\{x\in X : Ax\neq\phi\}$, $R(A)=\bigcup\{Ax : x\in D(A)\}$ and $|Ax|=\inf\{||y|| : y\in Ax\}$. For each set $D\subset X$, $\overline{D}$ represents the closure of D. For $x\in X$ and $r>0$, we denote by $B(x,r)$ the closed ball with center x and radius r.

DEFINITION 2.1. ([9]) *A continuous function $u:[0,T]\rightarrow D(A)\cap U$ with $u(0)=u_0$ is called an integral solution of* (E) *if*

(2.3)
$$||u(t)-x||^2\leqq||u(s)-x||^2 +2\int_s^t <F(u)(\theta)-y, u(\theta)-x>_s d\theta$$

for all $(x,y)\in A$ and $0\leqq s\leqq t\leqq T$.

Let $A\subset X\times X$ be an m-accreative set and $f\in L^1(0,T;X)$.

We recall that a function $u:[0,T]\rightarrow X$ is called a strong solution of the initial value problem

(2.4)
$$\frac{du(t)}{dt}+Au(t)\ni f(t),\ 0\leqq t\leqq T,$$

(2.5)
$$u(0)=u_0$$

if u is differentiable almost everywhere on $[0,T]$, absolutely continuous and satisfies $u(0)=u_0$ and $u'(t)+Au(t)\ni f(t)$ almost everywhere on $[0,T]$. It is well known that the initial value problem (2.4), (2.5) has a unique integral solution on $[0,T]$. Every strong solution of (2.4), (2.5) is also an integral solution of (2.4), (2.5) (see [1] [2]). Moreover, if u and v are two integral solution of (2.4), (2.5) corresponding to $f\in L^1(0,T;X)$ and $g\in L^1(0,T;X)$ respectively, then

(2.6) $||u(t)-v(t)||^2\leqq||u(s)-v(s)||^2$

$$+2\int_s^t < f(\theta)-g(\theta), u(\theta)-v(\theta)>_s d\theta$$

for all $0 \leqq s \leqq t \leqq T$. From (2.6), we also have that

(2.7) $$\|u(t)-v(t)\| \leqq \|u(s)-v(s)\| + \int_s^t \|f(\theta)-g(\theta)\| d\theta$$

for all $0 \leqq s \leqq t \leqq T$. For the proof, see [1], [2].

It is well known that each m-accretive set $A \subset X \times X$ generates a nonlinear semigroup of contrctions $S(t)$: $\overline{D(A)} \to \overline{D(A)}$, strongly continuous at the origin (see [1], [5]).

Using (2.7), we verify easily the following lemma (see also [10]).

LEMMA 2.1. *Let* $u:[0,T] \to \overline{D(A)}$ *be an integral solution of* (2.4), (2.5), $t \in [0,T)$, $s \in (0,T]$, $h \in R_+$ with $t+h \in [0,T]$ *and* $s-h \in [0,T]$. *Then, the following inequalities hold*:

(2.8) $$\|u(t)-u(t+h)\| \leqq \|S(h)u_0 - u_0\| + \int_0^h \|f(\theta)\| d\theta + \int_0^t \|f(\theta+h)-f(\theta)\| d\theta$$

and

(2.9) $$\|u(s)-u(s-h)\| \leqq \|S(h)u_0 - u_0\| + \int_0^h \|f(\theta)\| d\theta + \int_h^s \|f(\theta-h)-f(\theta)\| d\theta.$$

The following lemma has been obtained by Brezis [4].

LEMMA 2.2. *Let* $A \subset X \times X$ *be a m-accretive set and* $J_\lambda = (I+\lambda A)^{-1}$ *for* $\lambda > 0$. *Then, for every* $x \in \overline{D(A)}$, $t > 0$ *and* $\lambda > 0$, *we have the follwing inequality*

(2.10) $$\|x - J_\lambda x\| \leqq \left(1+\frac{\lambda}{t}\right)\frac{2}{t}\int_0^t \|x - S(\theta)x\| d\theta,$$

and in particular

(2.11) $$\|x - J_t x\| \leqq \frac{4}{t}\int_0^t \|x - S(\theta)x\| d\theta.$$

DEFINITION 2.2. *A mapping* $F: C(0,a;U) \to C(0,a;X)$ *is* *called* L^∞*-bounded if for each* $T\in(0,a]$, $v\in C(0,T;U)$ and $r>0$, *there exists* $g\in L^\infty(0,T;R_+)$ *such that*

(2.12) $\quad ||F(u)(t)||\leqq g(t)$ *a.e. on* $[0,T]$

for all $u\in C(0,T;U)$ *with* $||u(t)-v(t)||\leqq r$, *for* $0\leqq t\leqq T$.

We conclude this Section with following lemma.

LEMMA 2.3. *Let* F *be* L^∞*-bounded and* $T\in(0,a]$ *Then, for* $v\in C(0,T;U)$ *and* $r>0$, *the following relations*:

$$(2.13) \qquad \lim_{h\to 0^+}\int_0^t ||F(u)(s)-F(u)(s+h)||ds=0, \quad 0\leqq t<T,$$

$$(2.14) \qquad \lim_{h\to 0^+}\int_h^t ||F(u)(s)-F(u)(s-h)||ds=0, \quad 0<t\leqq T,$$

hold, uniformly with respect to all $u\in C(0,T;U)$ *with* $||u(t)-v(t)||\leqq r$ *for* $0\leqq t\leqq T$.

PROOF. This follows directly from Lebesgue convergence theorem.

3. Existence

THEOREM 3.1. *Assume that*

(C_1) X *is a real Banach space and* $U\subset X$ *is a given open set.*

(C_2) $A\subset X\times X$ *is a m-accretive set and* $(I+\lambda A)^{-1}$ *is compact for all* $\lambda>0$

(C_3) $F:C(0,a;U)\to C(0,a;X)$ *is continuous and* L^∞*-bounded.*

Then, for each $u_0\in\overline{D(A)}\cap U$, *there exists* $T\in(0,a]$ *such that problem* (E) *has at least one integral solution on* $[0,T]$.

REMARK 3.1. We note that in the case in which X is a real reflexive Banach space and $F(u)$ is of bounded variation, where $u\in C(0,a;U)$ is of bounded variation, the integral solution u provided by Theorem 3.1 is a strong solution (see [1] [7]), as well as, in the case in which X is a real reflexive Banach space, u is a weak solution (see [1]).

PROOF. Let $u_0 \in \overline{D(A)} \cap U$ and choose $T>0$, $r>0$ and $g \in L^\infty(0, T:R_-)$ such that

(3.1) $B(u_0, r) \subset \overline{D(A)} \cap U$,

(3.2) $\|F(u)(t)\| \leqq g(t)$ $a.e. on [oT]$

for all $u \in C(0, T;U)$ with $\|u(t)-u_0\| \leqq r$ for $0 \leqq t \leqq T$ and in addition,

(3.3) $\int_0^T g(s)ds + \|S(t)u_0 - u_0\| \leqq r$

for all $0 \leqq t \leqq T$. From condition (C_3) and continuity of $S(t)$ at the origin, we can choose such constants T, r and $g \in L^\infty(0, T;R_+)$. Now we set

(3.4) $K = \{u \in C(0, T;U); u(t) \in B(u_0, r)$ for all $0 \leqq t \leqq T\}$

and we deduce easily that K is nonempty, convex and closed in $C(0, T;X)$. Let $v \in K$. Then, by Benilan's existence and uniqueness theorem, the initial value problem

(3.5) $\dfrac{du(t)}{dt} + Au(t) \ni F(v)(t), \ 0 \leqq t \leqq T,$

(3.6) $u(0) = u_0$

has a unique integral solution $u \in C(0, T:X)$. Therefore we define the operator $Q: K \to C(0, T;X)$ by $Qv = u$, where u and v satisfy together (3.5), (3.6). Now let us observe that the problem (E) has at least one integral solution if and only if the operator Q has at least one fixed point. Thus it suffices to show the operator Q maps K into K and is completely continuous. To this end, we apply (2.7) to Qv and to the solution w of the problem

(3.7) $\dfrac{dw(t)}{dt} + Aw(t) \ni 0, \ 0 \leqq t \leqq T,$

(3.8) $w(0) = u_0$

Since the integral solution of (3.7), (3.8) is expressed as

(3.9) $w(t)=S(t)u_0,$

we obtain from (2.7)

(3.10) $$\|(Qv)(t)-S(t)u_0\| \leqq \int_0^t \|F(v)(s)\| ds.$$

Combining (3.10) with

(3.11) $$\|(Qv)(t)-u_0\| \leqq \|(Qv)(t)-S(t)u_0\| + \|S(t)u_0-u_0\|,$$

we get from (3.2)

(3.12) $$\|(Qv)(t)-u_0\| \leqq \int_0^t g(s)ds + \|S(t)u_0-u_0\|,$$

which in view of (3.3) implies

(3.13) $$\|(Qv)(t)-u_0\| \leqq r \text{ for all } 0 \leqq t \leqq T$$

But (3.13) show that $QK \subset K$. It is easy to show that Q is continuous on K in the uniform convergence topology (see [9]). To conclude that Q is completely continuous, we shall use Ascoli's theorem. First we show that the family $\{Qv : v \in K\}$ is equicontinuous on $[0, T]$. Using Lemma 2.1, we get

(3.14) $$\begin{aligned} &\|(Qv)(t)-(Qv)(t+h)\| \\ &\leqq \|S(h)u_0-u_0\| + \int_0^h \|F(v)(s)\| ds \\ &+ \int_0^t \|F(v)(s)-F(v)(s+h)\| ds \end{aligned}$$

for each $0 \leqq t < T$ and $h > 0$. From (3.2), we deduce

(3.15) $$\begin{aligned} &\|(Qv)(t)-(Qv)(t+h)\| \\ &\leqq \|S(h)u_0-u_0\| + \int_0^h g(s)ds \\ &+ \int_0^t \|F(v)(s)-F(v)(s+h)\| ds. \end{aligned}$$

In a similar manner we get, for each $0 < t \leqq T$ and $0 < h \leqq t$,

(3.16) $$\begin{aligned} &\|(Qv)(t)-(Qv)(t-h)\| \\ &\leqq \|S(h)u_0-u_0\| + \int_0^h g(s)ds \\ &+ \int_h^t \|F(v)(s)-F(v)(s-h)\| ds. \end{aligned}$$

Considering (2.13) and (2.14) in Lemma 2.3 and the continuity of the semigroup $S(t)$ at the origin, we obtain from

(3.15), (3.16) the equicontinuity of the family $\{Qv; v\in K\}$. Next we show that for each $0\leqq t\leqq T$, the set $\{(Qv)(t): v\in K\}$ is precompact in X. Obviously for $t=0$, the set above being a singleton $\{u_0\}$ is precompact. Then, let $0<t\leqq T$ and $0<\theta<t$, $v\in K$ and consider the following initial value problem

(3.17) $$\frac{dv^{\theta}(s)}{ds}+Av^{\theta}(s)\ni 0,\quad t-\theta\leqq s\leqq t+\theta,$$

(3.18) $$v^{\theta}(t-\theta)=(Qv)(t-\theta).$$

Applying (2.7) to v^{θ} and to Qv on $[t-\theta, t]$, we get

(3.19) $$\|(Qv)(t)-v^{\theta}(t)\|\leqq\int_{t-\theta}^{t}\|F(v)(r)\|dr.$$

From (3.2) and (3.19), it follows that

(3.20) $$\|(Qv)(t)-v^{\theta}(t)\|\leqq\int_{t-\theta}^{t}g(r)dr.$$

Now by the inequality (2.10) of Lemma 2.2, we get

(3.21) $$\|(Qv)(t)-J_{\lambda}(Qv)(t)\|$$
$$\leqq(1+\frac{\lambda}{s})\ \frac{2}{s}\int_{0}^{s}\|(Qv)(t)-S(\theta)(Qv)(t)\|d\theta.$$

Let us observe that

(3.22) $$\|(Qv)(t)-S(\theta)(Qv)(t)\|$$
$$\leqq\|(Qv)(t)-S(\theta)(Qv)(t-\theta)\|$$
$$+\|S(\theta)(Qv)(t-\theta)-S(\theta)(Qv)(t)\|$$

for all $0<\theta<t$. In view of the equality

(3.23) $$S(\theta)(Qv)(t-\theta)=v^{\theta}(t),$$

we get from (3.20) and (3.23)

(3.24) $$\|Qv(t)-S(\theta)(Qv)(t)\|$$
$$\leqq\|Qv(t)-v^{\theta}(t)\|+\|S(\theta)(Qv)(t-\theta)-S(\theta)(Qv)(t)\|$$
$$\leqq\int_{t-\theta}^{t}g(r)dr+\|(Qv)(t-\theta)-(Qv)(t)\|.$$

Using(3.24), we get from(3.21) for $\lambda=s\in(0,t)$,

(3.25) $$\|(Qv)(t)-J_{\lambda}(Qv)(t)\|$$
$$\leqq 4\int_{t-\lambda}^{t}g(r)dr+\frac{4}{\lambda}\int_{0}^{\lambda}\|(Qv)(t-\theta)-(Qv)(t)\|d\theta.$$

As family $\{Qv: v\in K\}$ is equicontinuous at each $0\leqq t\leqq T$, it follows that

(3.26) $\quad ||(Qv)(t-\theta)-(Qv)(t)||\leqq\varepsilon(\theta)$,

where $\varepsilon(h)$ is nondecreasing and $\varepsilon(h)\to 0_+$ as $h\to 0_+$. Thus from (3.25) and (3.26), we obtain

(3.27) $\quad ||(Qv)(t)-J_\lambda(Qv)(t)||\leqq 4(\int_{t-\lambda}^{t} g(r)dr+\varepsilon(\lambda))$.

Since $J_\lambda=(I+\lambda A)^{-1}$ is compact by (C_2) and $\{(Qv)(t); v\in K\}$ is bounded for all $0\leqq t\leqq T$, we conclude from (3.27) that the set $\{(Qv)(t): v\in K\}$ is precompact in X for all $0\leqq t\leqq T$. Therefore by Ascoli's theorem QK is relatively compact in $C(0, T; X)$, which complete the completely continuity of Q.

Finally, to prove Theorem 3.1 we have only to mention that the operator Q satisfies the hypotheses of Schauder's fixed point theorem, and thus Q has at least one fixed point $u\in K$, which is an integral solution of the problem (E).

COROLLARY 3.1. *Assume that X is a real finite dimensional Hilbert space, $A\subset X\times X$ is a m-accretive set and (C_3) is satisfied. Then, for each $u_0\in\overline{D(A)}\cap U$ there exists $T>0$ such that the problem* (E) *has at least one strong solution on* $[0, T]$.

PROOF. As X is finite dimensional and $A\subset X\times X$ is a m-accretive set, it follows that the operator $J_\lambda=(I+\lambda A)^{-1}$ is compact for all $\lambda>0$ and therefore we are in the hypotheses of Theorem 3.1. Now we have to remark that in the case in which X is finite dimensional Hilbert space, each integral solution is a strong solution (see [3]).

4. Continuation of solution, example

In this section, we shall give a result about the continuation of solution and an example which is an application of our results.

THEOREM 4.1. *Assume that* (C_1), (C_2) *and* (C_3) *are satisfied. Then, for each* $u_0 \in \overline{D(A)} \cap U$, *there exists an integral solution u of (E) defined on a maximal interval of existence* $[0, T_{max})$, *where either* $T_{max}=a$ *or if* $T_{max}<a$, *then* $\|u(t)\| \to \infty$ *as* $t \uparrow T_{max}$.

PROOF. Let $u(t)$ be an integral solution of problem(E) on $[0, t_1]$. We extend $u(t)$ to the interval$[0, t_1+\delta]$ with by defining

$$u(t+t_1)=w(t),$$

where $w(t)$ is an integral solution of the problem

$$\frac{dw(t)}{dt}+Aw(t) \ni F(w)(t), \tag{4.1}$$

$$w(0)=u(t_1). \tag{4.2}$$

From Theorem 3.1, there exists an integral solution $w(t)$ on an interval of positive length $\delta>0$. Let $[0, T_{max})$ be the maximal interval to which the integral solution $u(t)$ of problem (E) can be extended. Now we shall prove that if $T_{max}<a$, them $\|u(t)\| \to \infty$ as $t \uparrow T_{max}$. First we show that if $T_{max}<a$, then $\limsup_{t \uparrow T_{max}} \|u(t)\| = \infty$. To this end, assume that if $T_{max}<a$, $\limsup_{t \uparrow T_{max}} \|u(t)\| < \infty$. Then there exists $K>0$ such that for $0 \leqq t < T_{max}$,

$$\|u(t)\| \leqq K. \tag{4.3}$$

and by our assumption on function F, there exists $g \in L^{\infty}(0, T_{max}; R_+)$ such that

$$\|F(u)(t)\| \leqq g(t) \text{ a.e. on } [0, T_{max}]. \tag{4.4}$$

If $0<t<t'<T_{max}$, then we have from (2.6) that

$$\begin{aligned} \|u(t')-u(t)\|^2 &\leqq 2\int_t^{t'} \langle F(u)(\theta)-y, u(\theta)-u(t) \rangle_s d\theta \\ &\leqq 2\int_t^{t'} \|F(u)(\theta)-y\| \|u(\theta)-u(t)\| d\theta \end{aligned} \tag{4.5}$$

$$\leqq 4K\int_t^{t'}(\|Fu(\theta)\|+|Au(t)|d\theta$$
$$\leqq 4K[\|g\|_{L^\infty(0,T_{max};R_+)}(t'-t)$$
$$+\int_t^{t'}|Au(\theta)|d\theta],$$

where $(u(t),y)\in A$. From (4.5) we get $\|u(t')-u(t)\|\to 0$ as t', $t\uparrow T_{max}$. Thus $\lim_{t\uparrow T_{max}} u(t)=u(T_{max})$ exists and by the first part of the proof, the solution u can be extended beyond T_{max}, contradicting the maximality of T_{max}. Therefore the assumption $T_{max}<a$ implies $\lim_{t\uparrow T_{max}}\sup\|u(t)\|=\infty$. To complete the proof, we show that $\|u(t)\|\to\infty$ as $t\uparrow T_{max}$. If it is false, then there exist $K>0$ and a sequence $\{t_n\}$ such that $t_n\ T_{max}$ and for all n and $x\in D(A)$,

(4.6) $\|u(t_n)-x\|^2\leqq K.$

We also deduce that

(4.7) $\|F(u)(t)\|+|Ax|\leqq g(t)+M$ a.e. on $[t, T_{max}]$

for some $g\in L^\infty(0, T_{max};R_+)$ and some $M>0$. Since $t\to \|u(t)-x\|$ is continuous and $\lim_{t\uparrow T_{max}}\sup\|u(t)\|=\infty$, there exists a sequence $\{h_n\}$ with the following properties: $h_n\to 0$ as $n\to\infty$,

(4.8) $\|u(t)-x\|^2\leqq K+1$

for $t_n\leqq t\leqq t_n+h_n$, and

(4.9) $\|u(t_n+h_n)-x\|^2=K+1.$

Then we have from (2.6)

(4.10) $K+1=\|u(t_n+h_n)-x\|^2$

$$\leqq\|u(t_n)-x\|^2+2\int_{t_n}^{t_n+h_n}<F(u)(\theta)-y, u(\theta)-x>_s d\theta$$
$$\leqq K+2\int_{t_n}^{t_n+h_n}\|F(u)(\theta)-y\|\ \|u(\theta)-x\|d\theta$$
$$\leqq K+2\sqrt{K+1}(\|g\|_{L^\infty(0,T_{max};R_+)}+M)h_n,$$

where $(x,y)\in A$. This give a contraction as $n\to\infty$ Thus we conclude that $\|u(t)\|\to\infty$ *as* $t\uparrow T_{max}$, which completes the proof of Theorem 4.1.

We note that each continuous function $f:[0,a]\times U\to X$ generates a unique continuous function $F:C(0,a;U)\to C(0,a;X)$. Thus we obtain the following.

COROLLARY 4.1. *Assume that* (C_1) *and* (C_2) *are satisfied and in addition that* $f:[0,\infty)\times U\to X$ *is a continuous mapping. Assume further that*

(4.11) $\|f(t,x)\|\leqq K_1(t)\|x\|+K_2$

for each $(t,x)\in[0,\infty)\times U$, *where* $K_1\in L^\infty_{loc}(0,\infty;R)$ *and* $K_2\in R$. *Then, for each* $u_0\in\overline{D(A)}\cap U$, *there exists an integral solution of* (E) *defined on the whole positive half axis.*

PROOF. From(4.11), it is easy to show that a function F generated by f is L^∞-bounded, and hence, by Theorem 4.1, the Corollary follows.

EXAMPLE 4.1. Let Ω be a bounded open subset of R^n with sufficiently smooth boundary Γ. Let $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $H^k(\Omega)$ and $H_0^k(\Omega)$ stand for Sobolev spaces on Ω.

We consider a nonlinear differential operator of the form

(4.12) $$Au=\sum_{|\alpha|\leqq m}(-1)^{\alpha}D^{\alpha}A_{\alpha}(x,u,Du,\dots,D^m u),$$

where $A_\alpha(x,z)$ are real functions defined on $\Omega\times R^m$ and satisfy the following conditions:

(I) A_α are measurable x and continuous in z for all α. There exist $p>1$, $g\in L^q(\Omega)$ $(\frac{1}{p}+\frac{1}{q}=1)$ and a positive constant C such that

(4.13) $|A_\alpha(x,z)|\leqq C(|z|^{p-1}+g(x))$ a.e. $x\in\Omega$.

(II) For any $(y,z)\in R^m\times R^m$ and for almost every $x\in\Omega$, the following inequality holds:

(4.14) $$\sum_{|\alpha|\leq m}(A_\alpha(x,z)-A_\alpha(x,y))(z_\alpha-y_\alpha)\geqq w(\sum_{|\alpha|\leq m}|z_\alpha-y_\alpha|^2),$$

where $w>0$.

We recall that the operator $A: W_0^{m,p}(\Omega)\to W^{-m,p}(\Omega)$ defined by

(4.15) $$(Au,v)=\sum_{|\alpha|\leq m}\int_\Omega A_\alpha(x,u,Du,\ldots,D^m u)D^\alpha v dx$$

for $u,v\in W_0^{m,q}(\Omega)$, is monotone and demicontinuous and

(4.16) $$||Au||_{-m,q}\leqq C(1+||u||_{m,p}^{p-1})$$

(see [1]).

Now we consider the nonlinear boundary value problem of the parabolic type

(4.17) $$\frac{\partial u}{\partial t}+\sum_{|\alpha|\leq m}(-1)^\alpha D^\alpha A_\alpha(x,u,Du,\ldots,D^m u)=f$$

with Dirichlet boundary conditions

(4.18) $D^\alpha u=0$ on $[0,T]\times\Gamma$ for $|\alpha|\leqq m-1$

and initial condition

(4.19) $u(0,x)=u_0(x)$ on Ω.

THEOREM 4.2. *Let $H=L^2(\Omega)$, $V=H_0^m(\Omega)$ and $A: H_0^m(\Omega)\to H^{-m}(\Omega)$ be the nonlinear operator defined above.* Let *$f:[0,\infty)\times H\to H$ be a continuous function Then, for $u_0\in H$, there exists $T>0$ such that* (4.17), (4.18), (4.19) *has at least one integral solution on* $[0,T]$.

PROOF. Let A_H be a operator defined by

(4.20) $$A_H u=Au$$

for $u\in D(A_H)=\{u\in V: Au\in H\}$. Then A_H is a m-accretive operator on H (see [1], [9]). Now let us remark that (4.17), (4.18), (4.19) can be rewritten in the form

(4.21) $$\frac{du(t)}{dt}+A_H(t)=F(u)(t),$$

(4.22) $u(0)=u_0,$

where

(4.23) $F(u)(t)=f(t, u(t)).$

Since A_H is coercive by (II) and the inclusion mapping from V into H is completely continuous, we can show that $(I+\lambda A_H)^{-1}$ is compact for $\lambda>0$. It is easy to see that for each $T>0$, $F: C(0, T; H)\to C(0, T; H)$ is continuous. Since $f(t, u(t))\in H$ for any $u\in C(0, T: H)$, it is obvious that F is L^∞-bounded. Therefore by applying Theorem 3.1, we obtain the conclusion of Theorem. 4.2.

References

[1] V. Barbu, Nonlinear Semigroup and Differential Equations in Banach Spaces, Editura Academiei, Bucurresti-Noordhoff, 1976.

[2] P. Benilan, Equations D'évolution dans une espace de Banach quelconque et applications, Thése de Doctorat d'État, Univ. de Paris (Orsay), 1972.

[3] P. Benilan and H. brezis, Solutions faibles d'équations d'évolution dans les espaces de Hilbert, Ann. Inst. Fouries, 22 (1972), 311-329.

[4] H. Brézis, New results concerning monotone operators and nonlinear semigroup, in Lecture Notes of the Reserch Institute for Mathematical Sciences, Kyoto Univ., 258 (1975), 2-27.

[5] M.G. Crandall and T. Liggett, Generation of semigroups of nonlinear transformations in general Banach spaces, Amer. J. Math., 93 (1971), 265-298.

[6] M.G. Crandall and J. Nohel, An abstract functional differential equation and a related Volterra equation, Israel J. Math., 29 (1978), 313-328

[7] N. Hirano, Local existence theorem for nonlinear Differential equations, SIAM J. Math. Anal. 14 (1983), 117-125.

[8] A. Pazy, A class of semi-linear equations of evolution, Israel

J. Math., 20 (1975), 23-36

[9] I.I. Vrabie, The nonlinear version of Pazy's local existence theorem, Israel J. Math., 32 (1979) 221-235.

[10] I.I. Vrabie, Compactness methods for an abstract nonlinear Volterra integrodifferential equations, Nonlinear Analysis Theory, Methods and Appl., 5 (1981), 355-371.

Department of Mathematics
Dong-A University
Pusan 600-02
Korea