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# A COMMON FIXED POINT THEOREM FOR TWO SYSTEMS OF TRANSFORMATIONS

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*Dedicated to the memory of late Professor S.R. Sinha*

With a view to generalizing the well-known Banach contraction principle, in 1973, Matkowski gave a fixed point theorem for a system of $n$ transformations, each from a product of $n$ metric spaces to one of the spaces. In this note, a common fixed point theorem for two systems of transformations generalizing the results of Păvăloiu(1969), Rus (1972) and Matkowski (1973) is proved. Our theorem also includes the result of Czerwik (1976a) for a system of single-valued transformations under a slightly different condition.

We shall follow the notations of Matkowski (1973, 1975).

$$(1)\quad c_{ik}^{(0)}=\begin{cases} a_{ik} & \text{for } i\neq k,\ i,k=1,\cdots,n,\\ 1-a_{ik} & \text{for } i=k,\end{cases}$$

and $c_{ik}^{(l)}$ are defined recursively by

$$(2)\quad c_{ik}^{(l+1)}=\begin{cases} c_{11}^{(l)}c_{i+1,k+1}^{(l)}+c_{i+1,1}^{(l)}c_{1,k+1}^{(l)}, & \text{for } i\neq k,\\ c_{11}^{(l)}c_{i+1,k+1}^{(l)}-c_{i+1,1}^{(l)}c_{1,k+1}^{(l)}, & \text{for } i=k,\end{cases}$$

$$i,k=1,\cdots,n-l-1,\quad l=0,1,\cdots,n-2.$$

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We remark that $(c_{ik}^{(l)})$ is an $(n-l)\times(n-l)$ matrix.

Following Matkowski's Lemma (1973) or Lemma 1.1 (1975, p. 9) and the beginning part of the proof of Theorem 1.4 (Matkowski 1975, p. 11), we have

LEMMA 1. *Let* $c_{ik}^{(o)}>0$ *for* $i,k=1,\cdots,n,\ n\geq 2$. *The system of inequalities*

$$\sum_{k=1}^{n} a_{ik}r_k < r_i,\quad i=1,\cdots,n, \tag{3}$$

*has a positive solution* $r_1, r_2, \cdots, r_n$ *if and only if the following inequalities hold*:

$$c_{ii}^{(l)}>0,\quad i=1,2,\cdots,n-1,\quad l=0,\cdots,n-1. \tag{4}$$

Throughout this note let $(X_i, d_i)$ be complete metric spaces and $X_1\times X_2\times\cdots\times X_n=X$.

Matkowski (1973) announced, in the Bulletin of the Polish Academy of Sciences, a contraction principle, in fact a generalization of the well-known Banach contraction principle, for a system of transformations from $X$ to $X_i$, $i=1,\cdots,n$ (see Remark 3 below. Its proof appears in (Matkowski 1975), and, in some circles, this is now called Matkowski's contraction principle (MCP). Following the MCP, Czerwik (1976a) proved a fixed point theorem for a system of multi-valued transformations which includes among others the contraction principle for multi-valued transformations (Nadler, 1969) and the MCP. Moreover, utilizing the proof-technique of Matkowski (1975), Czerwik (1976b) and Reddy and Subrahmanyam (1981) generalized, respectively, Edelstein's fixed point theorem (Edelstein 1962) and Krasnoselskii's fixed point theorem for the sum of two transformations (Krasnoselskii 1955). Matkowski type fixed point theorems are applicable to convex solutions of a system of functional equations (Reddy and Subrahmanyam 1981). We combine, in this note, the ideas of Matkowski (1973, 1975), Wong (1973) and Czerwick (1976a), and prove a common fixed point theorem for two systems of transformations from $X$ to $X_i$, $i=1,\cdots,n$. Several fixed point theorems follow as corollaries to our result (Theorem 2).

We prove the following result.

THEOREM 2. *Let* $F_i, G_i : X \to X_i$, $i=1,2,\cdots,n$ *be two sets of transformations. If there exist nonnegative numbers* $b$ *and* $a_{ik}$, $i,k=1,2,\cdots,n$ *such that*

$$(5)\ d_i(F_i(x_1,\cdots,x_n), G_i(z_1,\cdots,z_n))$$
$$\leq \sum_{k=1}^{n} a_{ik} d_k(x_k, z_k)$$
$$+b[d_i(x_i, F_i(x_1,\cdots,x_n))+d(z\ G_i(z_1,\cdots,z_n))]$$

*for all* $x_j, z_j \in X_j$, $i,j=1,2,\cdots,n$,

*and the numbers* $c_{ik}^{(o)}$ *and* $c_{ik}^{(1)}$ *defined by* (1) *and* (2) *fulfil the condition* (4), *where*

$$(6)\ 0 \leq 2b < 1-\nu,\ \text{in which}\ \nu = \max\,(r_i^{-1}\sum_{k=1}^{n} a_{ik} r_k),$$

*then the system of equations*

$$F_i(x_1,\cdots,x_n)=x_i=G_i(x_1,\cdots,x_n)$$

*has a unique common solution* $x_1,\cdots,x_n$ *such that* $x_i \in X_i$, $i=1,\cdots,n$.

PROOF. First of all we note that, in view of the homogeneity of the system of inequalities (3), $\nu$ defined in (6) exists and $0<\nu<1$ (Czerwik 1976a). From Lemma 1 and (6) we may choose a system of positive numbers $r_1, r_2,\cdots,r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq \nu r_i, \quad i=1, \cdots, n.$$

Pick $x_i^0 \in X_i$ and choose a sequence $\{x_i^m\}$ in $X_i$, $i=1, \cdots, n$ such that

(7) $x_i^{2m+1} = F_i(x_1^{2m}, \cdots, x_n^{2m})$ and $x_i^{2m+2} = G_i(x_1^{2m+1}, \cdots, x_n^{2m+1})$, $m=0, 1 \cdots$.

Without loss of generality (in fact, increasing the numbers $r_i$ by a constant multiple, if necessary), we may assume that

$$d_i(X_i^0, x_i^1) \leq r_i, \quad r_i \geq 1 \text{ for } i=1, \cdots, n.$$

By (5) and (7),

$$d(x_i^1, x_i^2) = d_i(F_i(x_1^0, \cdots, x_n^0), G_i(x_1^1, \cdots, x_n^1))$$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(x_k^0, x_k^1)$$

$$+ b[d_i(x_i^0, F_i(x_1^0, \cdots, x_n^0)) + d_i(x_i^1, G_i(x_1^1, \cdots, x_n^1))]$$

$$\leq \sum_{k=1}^{n} a_{ik} r_k + b[d_i(x_i^0, x_i^1) + d_i(x_i^1, x_i^2)].$$

So $d_i(x_i^1, x_i^2) \leq q r_i$, $i=1, \cdots, n$, where $0 < q = (\nu + b)/(1-b) < 1$.

Also from

$$d_i(x_i^2, x_i^3) = d_i(F_i(x_1^2, \cdots, x_n^2), G_i(x_1^1, \cdots, x_n^1))$$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(x_k^1, x_k^2)$$

$$+ b[d_i(x_i^2, F_i(x_1^2, \cdots, x_n^2)) + d_i(x_i^1, G_i(x_1^1, \cdots, x_n^1))]$$

$$\leq q \sum_{k=1}^{n} a_{ik} r_k + b[d_i(x_i^2, x_i^3) + d_i(x_i^1, x_i^2)],$$

we obtain

$$d_i(x_i^2, x_i^3) \leq q^2 r_i, \quad i=1, \cdots, n.$$

Inductively,

$$d_i(x_i^m, x_i^{m+1}) \leq q^m r_i, \quad i=1, \cdots, n. \tag{8}$$

It follows easily from (8) that $\{x_i^m\}$ is a Cauchy sequence for each $i=1, \cdots, n$. Since $(X_i, d_i)$ is complete, $\{x_i^m\}$ converges to a point $u_i$ in $X_i$ such that

$$u_i = \lim_{m \to \infty} x_i^m, \quad i=1, \cdots, n.$$

Now we show that

$$u_i = F_i(u_1, \cdots, u_n) = G_i(u_1, \cdots, u_n), \quad i=1, \cdots, n.$$

For each $1 \leq i \leq n$,

$$\begin{aligned} d_i[u_i, F_i(u_1, \cdots, u_n)] &\leq d_i(u_i, x_i^{2m+2}) + d_i(x_i^{2m+2}, F_i(u_1, \cdots, u_n)) \\ &= d_i(u_i, x_i^{2m+2}) \\ &\quad + d_i(G_i(x_1^{2m+1}, \cdots, x_n^{2m+1}), F_i(u_1, \cdots, u_n)) \\ &\leq d_i(u_i, x_i^{2m+2}) + \sum_{k=1}^{n} a_{ik} d_k(u_k, x_k^{2m+1}) \\ &\quad + b[d_i(u_i, F_i(u_1, \cdots, u_n)) \\ &\quad + d_i(x_i^{2m+1}, G_i(x_i^{2m+1}, \cdots, x_n^{2m+1}))] \\ &\leq d_i(u_i, x_i^{2m+2}) + \sum_{k=1}^{n} a_{ik} d_k(u_k, x_k^{2m+1}) \\ &\quad + b[d_i(u_i, F_i(u_1, \cdots, u_n)) + d_i(x_i^{2m+1}, x_i^{2m+2})]. \end{aligned}$$

Making $m \to \infty$, we obtain for each $0 \leq i \leq n$,

$$(1-b) d_i(u_i, F_i(u_1, \cdots, u_n)) \leq 0,$$

implying

$$u_i = F_i(u_1, \cdots, u_n).$$

Similarly,

$$u_i = G_i(u_1, \cdots, u_n), \quad i=1, \cdots, n.$$

To prove the uniqueness of $u_i$, let $u_i$, $\bar{u}_i$ be two distinct solutions. Then

$$u_i = F_i(u_1, \cdots, u_n) = G_i(u_1, \cdots, u_n)$$

and

$$\bar{u}_i = F_i(\bar{u}_1, \cdots, \bar{u}_n) = G_i(\bar{u}_1, \cdots, \bar{u}_n).$$

We can assume that $d_i(u_i, \bar{u}_i) \le r_i$, $i = 1, \cdots, n$. So

$$\begin{aligned} d_i(u_i, \bar{u}_i) &= d_i(F_i(u_1, \cdots, u_n), G_i(\bar{u}_1, \cdots, \bar{u}_n)) \\ &\le \sum_{k=1}^{n} a_{ki} d_k(u_k, \bar{u}_k) + b[d_i(u_i, F_i(u_1, \cdots u_n)) \\ &\qquad + d_i(\bar{u}_i, G_i(\bar{u}_1, \cdots, \bar{u}_n))] \\ &= \sum_{k=1}^{n} a_i^k d_k(u_k, \bar{u}_k) \\ &\le \sum_{k=1}^{n} a_{ik} r_k \le \nu r_i. \end{aligned}$$

Inductively,

$$d_i(u_i, \bar{u}_i) \le \nu^m r_i, \ i = 1, \cdots, n.$$

Therefore $d_i(u_i, \bar{u}_i) = 0$, $i = 1, \cdots, n$. This completes the proof.

REMARK 3. In case $F_i = G_i$, $i = 1, 2, \cdots, n$ and $b = 0$, the result of Matkowski (1973 or 1975, Th. 1. 4) is obtained as a special case of Theorem 1. The Banach contraction principle is obtained by taking $n = 1$, $F_i = G_i$ and $b = 0$. For $n = 2$, $F_i = G_i$ and $b = 0$, we obtain the results of Păvăloiu (1969) and Rus (1972). The result of Czerwik (1976a) for a system of singlevalued transformations is obtained by taking $F_i = G_i$ in Theorem 2, under a slightly different condition.

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