THE JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
19: 11～14, 1986

ASTIGMATIC PROPERTY OF N-BODY GRAVITATIONAL LENS

CHANG, Kyongae
Department of Physics and Optical Engineering, Chongju University
(Received May 29, 1986)

ABSTRACT

It is shown in this paper that the astigmatic property of single gravitational lens in static bounded gravitational field can be retained, if n-gravitating body as a whole acts simultaneously as gravitational lens.

I. INTRODUCTION

It is well known that due to the gravitational lens effect the rays (normal to the wavefront) will not converge to a single point. Each ray bent upon passing a spherical mass M with impact parameter b will converges, however, at its own focal length (its own caustic, respectively) (Rafsdal 1964). Such an aberration is called astigmatism in geometrical optics. The astigmatic property of a single gravitational lens can be easily seen from the definition of focal length of gravitational lens.

For a deflected light ray by a gravitating body, the focal length is given by

$$f=\frac{b}{\Theta}, \tag{1}$$

where b is the impact parameter, and Θ, Einstein bending angle,

$$\Theta=\frac{4\ G\ M}{c^2 b} \tag{2}$$

We will show that n gravitating bodies lying in the same plane, which act as gravitational lenses, also have such an astigmatic behaviour. In this paper we will not discuss the focal length to the anti-caustic.

II. BENDING ANGLE AND ASTIGMATIC PROPERTY OF A SINGLE GRAVITATIONAL LENS

An astigmatic lens has, as well known in geometrical optics, two mutually perpendicular focal lines l_1 and l_2 at the focal distances f_1 and f_2, respectively.

Let us introduce an orthogonal coordinate system ξ', and η', setting the origin of the ξ'-η' coordinate at the centre of the lens. We now choose the orientation of ξ', η' axes parallel to the focal lines l_1 and l_2, respectively. Then, the angular deflection of a light ray with the impact parameter b (ξ', η') is given by

$$\begin{aligned} \Theta_{\xi 1} &= -K_1 \xi', \\ \Theta_{\eta 1} &= -K_2 \eta' \end{aligned} \tag{3}$$

where

$$K_i = f_i^{-1}; \qquad i = 1,\ 2 \tag{4}$$

and f_i is the focal distance of the deflected light ray with respect the focal line l_i.
Equation (3) may be rewritten in matrix as follows:

$$\begin{pmatrix} \Theta_{\xi'} \\ \Theta_{\eta'} \end{pmatrix} = \begin{pmatrix} -K_1 & 0 \\ 0 & -K_2 \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} \tag{5}$$

Now, let us assume an additional orthogonal coordinate system ξ and η with the same origin as before. By a rotation with an angle φ, let the ξ-η system merge into the ξ'-η' system.

We will now describe the angular deflection of an astigmatic lens rotated by an angle φ with respect to ξ and η coordinate system. We denote a vector in the plane of the lens where the gravitating body is projected as follows:

$$\mathbf{A}' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$

for the ξ'-η' coordinate system,
while

$$\mathbf{A} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

for the ξ-η coordinate system.
The relation between $\mathbf{A}'$ and $\mathbf{A}$ is then

$$\mathbf{A}' = \mathbf{D}\mathbf{A}, \tag{6}$$

where

$$\mathbf{D} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{7}$$

Hence,

$$\begin{pmatrix} \Theta_{\xi'} \\ \Theta_{\eta'} \end{pmatrix} = \mathbf{D} \begin{pmatrix} \Theta_{\xi} \\ \Theta_{\eta} \end{pmatrix} \tag{8}$$

Substituting equations (6) and (8) into equation (5), we have

$$\begin{pmatrix} \Theta_\xi \\ \Theta_\eta \end{pmatrix} = \mathbf{MA}, \tag{9}$$

where

$$\mathbf{M} = \mathbf{D}^{-1} \begin{pmatrix} -K_1 & 0 \\ 0 & -K_2 \end{pmatrix} \mathbf{D} \tag{10}$$

With eq. (7), we have

$$\mathbf{M} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} \tag{11}$$

where

$$\alpha = -\frac{K_1+K_2}{2}, \quad \beta = -\frac{K_1-K_2}{2}$$

Equation (9) represents the angular deflection of an astigmatic lens rotated by an angle φ

III. APPLICATION TO N-GRAVITATING BODIES AS GRAVITATIONAL LENS

In section 2 we have seen astigmatic property of a single gravitational lens. We shall show the identical behaviour of n discrete gravitating bodies, when they act as a compound gravitational lens.

For the purpose, we take n galaxies as gravitational lenses, i.e. n astigmatic lenses. We combine them together in such a way that the origin of their orthogonal axis should be all at the identical origin of the coordinate (ξ, η) in the plane where n galaxies are projected. Then, the total angular deflection contributed by n galaxies is given by Chang(1981)

$$\sum_{i=1}^{n} \boldsymbol{\theta}_i = \sum_{i=1}^{n} M_i\, A = \left(\sum_{i=1}^{n} M_i\right) A \; ; \qquad i = 1, 2, \ldots, n. \tag{12}$$

From equation. (11)

$$\sum_{i=1}^{n} M_i = -\begin{pmatrix} \sum_{i=1}^{n} \left(\frac{K_1+K_2}{2}\right)_i & 0 \\ 0 & \sum_{i=1}^{n} \left(\frac{K_1+K_2}{2}\right)_i \end{pmatrix} -$$

$$\begin{pmatrix} \sum_{i=1}^{n} \left(\frac{K_1-K_2}{2}\right) \cos 2\varphi_i & \sum_{i=1}^{n} \left(\frac{K_1-K_2}{2}\right) \sin 2\varphi_i \\ \sum_{i=1}^{n} \left(\frac{K_1-K_2}{2}\right) \sin 2\varphi_i & -\sum_{i=1}^{n} \left(\frac{K_1-K_2}{2}\right) \cos 2\varphi_i . \end{pmatrix} \tag{13}$$

Hence, from equations (9), (12), and (13) we have

$$\begin{pmatrix} \tilde{\theta}_\xi \\ \tilde{\theta}_\eta \end{pmatrix} = \left(-a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - b \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} \right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{14}$$

where

$$\begin{aligned} \tilde{\theta}_\xi &= \sum_{i=1}^{n} \theta_{\xi_i} \\ \tilde{\theta}_\eta &= \sum_{i=1}^{n} \theta_{\eta_i} \end{aligned} \tag{15}$$

and

$$a = \sum_{i=1}^{n} \left(\frac{K_1 + K_2}{2} \right)_i$$

$$b = \sqrt{ \left(\sum_{i=1}^{n} \left(\frac{K_1 - K_2}{2} \right)_i \cos 2\varphi_i \right)^2 + \left(\sum_{i=1}^{n} \left(\frac{K_1 - K_2}{2} \right) \sin 2\varphi_i \right)^2 } \tag{16}$$

We note that in equation (14) $\tilde{\theta}_\xi$ and $\tilde{\theta}_\eta$ are the sum of the bending angles contributed by n gravitational lenses. Equation (14) shows that the compound gravitational lens consisting of n gravitating bodies with static bounded gravitational field will also have the astigmatic property just as a single gravitational lens.

REFERENCES

Refsdal, S. 1964, M. N. R. A. S., **128**, 295.
Chang, K. 1981, Ph. D. Dissertation, Hamburg University.