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A Continuous Review (s, S) Inventory Model in which Depletion is Due to Demand and Loss of Units

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Abstract

A stochastic model for an inventory system in which depletion of stock takes place due to random demand as well as random loss of items is studied under the assumption that the intervals between successive unit demands, as well as those between successive unit losses are independently and identically distributed random variables having negative exponential distribution with respective parameters.

We have derived the steady state probability distribution of the stock level assuming instantaneous delivery of order under (s, S) inventory policy.

Also we have derived total expected cost expression and the necessary conditions to be satisfied for an optimal solution.

1. INTRODUCTION

Innumerable papers have been written analyzing mathematical models for describing ordering policies for one or more products. Invariably, it was assumed implicitly that once units enter into inventory, they "live" forever or else they expire after only a single planning period. Though this assumption is not

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altogether unreasonable or unjustifiable in cases where planning over a short run is essentially unaffected by obsolescence, there is a wide variety of situations that arise in the food industry, drug industry and in the area of health administration in which perishable nature of the inventory has to be taken into account in developing optimal ordering policies.

Some researchers have considered this aspect of the inventory [3], [4] and [5] but have assumed that the items, like foodstuffs, photographic films, drugs and pharmaceuticals and blood, have fixed life time.

But there exist a significant class of problems for which the life time of the items in the inventory is not deterministic and the assumption of random life time of the items would be more appropriate (e.g. the electronic industry).

Recently, Kumaraswamy and Sankarasubramanian [2] have considered a problem in which the depletion of the inventory level is due to random demand as well as random failure of items. But in their paper they assumed that perishability occurs only for one unit. Perishability in radioactive substances, spoilage in food grain storage, breakage in glasswares, and pilferage from on hand inventory, ect are nearly proportional to the on hand inventory.

In this paper, a similar problem is considered in which the depletion of inventory level is due to random demand as well as random loss of units proportional to on hand inventory.

We have assumed that demands occur in a poisson manner with parameter μ and that life time distribution of an item in inventory is negative exponential with parameter λ.

We have derived the steady state probability distribution of the stock level assuming instantaneous delivery of order under (s, S) inventory policy.

Also we have derived total expected cost expression and the necessary conditions to be satisfied for an opitmal solution.

2. ANALYSIS OF THE PROBLEM

We consider the problem in which the maximum inventory level is fixed as S units.

As soon as the stock level drops to s, an order is placed for a constant quantity Q=S-s to bring the level S. We assume that the replenishment of the stock takes place instantaneously. Demands for the items occur in a Poisson manner with parameter μ, and the life time of an item is negative exponential with parameter λ. Thus, the decrease in the stock level is due to either a demand or a loss of an item.

Figure 1 is a realization of the stochastic process generated by the movement of stock level on hand inventory over time.

Let H(t) be the stock level at an arbitrary time t. Let $P(n, t) = P\{H(t) = n\}$, n = s+1, ..., S, denote the probability that there are exactly n units in stock at time t.

In a time dt, the inventory level moves from state n+1 to state n if a demand or a loss occurs.

The inventory level stays at n if neither a demand nor a loss occur. If the inventory level is in state s+1 and a demand or a loss occurs, then inventory level moves to state S since the replenishment of the stock takes place instantaneously.

Thus we have the diagram representing the transitions shown in Figure 2. The following equations represent the transistions shown in Figure 2.

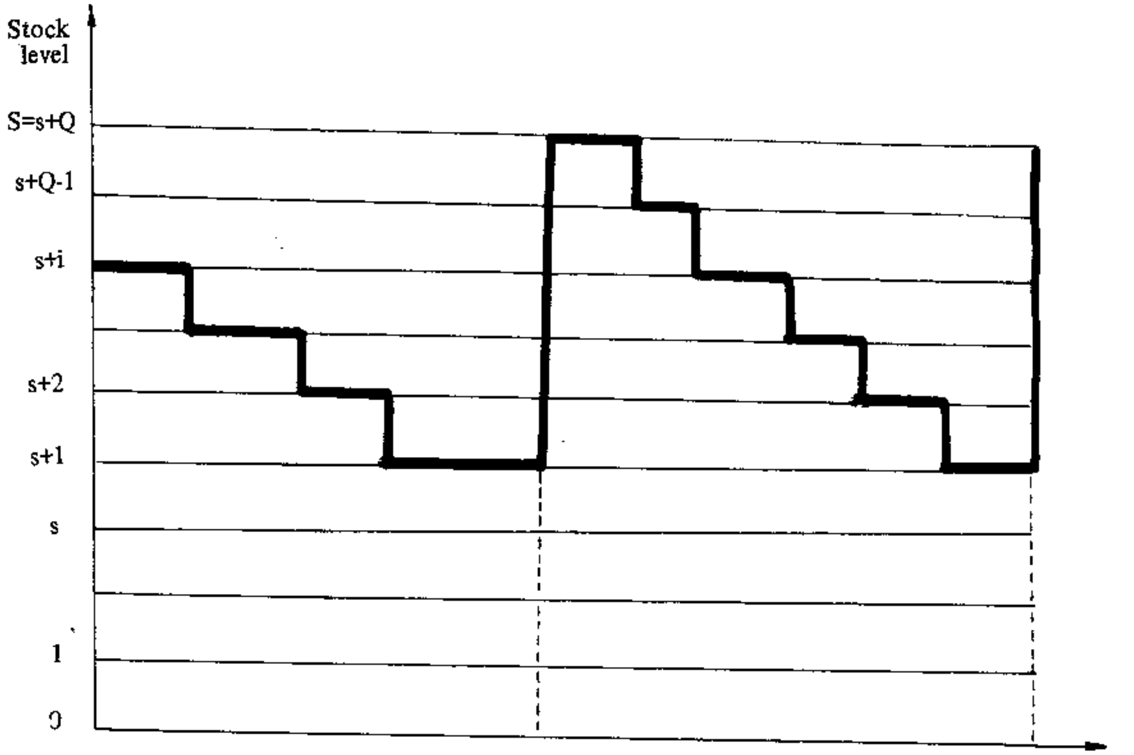

Figure 1. Geometry of the inventory system operating under the (s, S) policy.

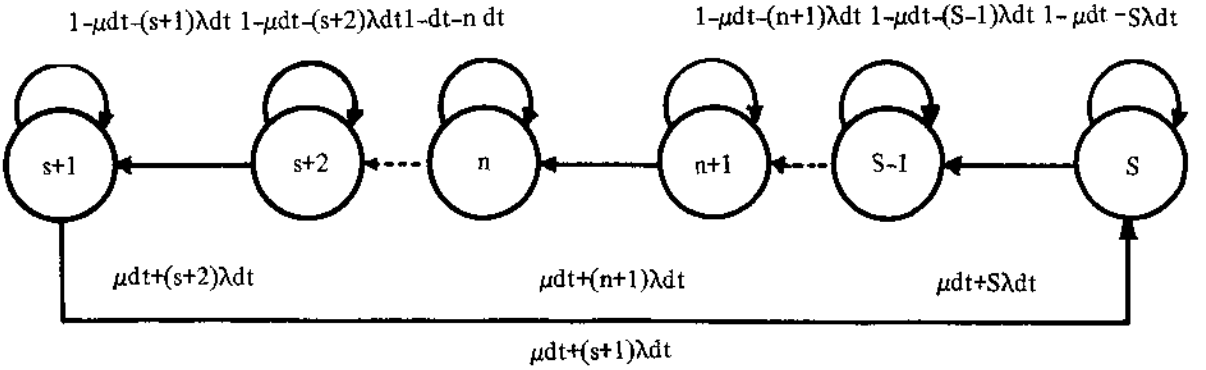

Figure 2. Transition diagram.

For $s+1 \leqslant n \leqslant S-1$ and $t \geqslant 0$,

$$p(n, t+dt) = P(n, t)[1-\mu dt-n\lambda dt+0(dt)] + P(n+1, t)[\mu dt+(n+1)\lambda dt+0(dt)].$$

For $n = S$ and $t \geqslant 0$,

$$p(S, t+dt) = P(S, t)[1-\mu dt-S\lambda dt+0(dt)] + P(s+1, t)[\mu dt+(s+1)\lambda dt+0(dt)].$$

Transposing, dividing by dt, and letting $dt \to 0$, these equations can be written as

$$\frac{dp(n, t)}{dt} = -(\mu+n\lambda) P(n, t) + [\mu+(n+1)\lambda]P(n+1, t), \qquad s+1 \leqslant n \leqslant S-1 \tag{1}$$

$$\frac{dp(S, t)}{dt} = -(\mu+S\lambda) P(S, t) + [\mu+(s+1)\lambda]P(s+1, t). \tag{2}$$

Let P_n be the probability that in the steady state there are exactly n units in stock, $n = s+1, \ldots\ldots, S$, that is

$$P_n = \lim_{t\to\infty} P(n, t) = \lim_{t\to\infty} P\ \{H(t) = n\}.$$

Nothing that $S = s+Q$, the following steady state equations are obtained from (1) and (2);

$$-(\mu+n\lambda) P_n + [\mu+(n+1)\lambda]\ P_{n+1} = 0, \qquad s+1 \leqslant n \leqslant s+Q-1 \tag{3}$$

$$-(\mu+S\lambda) P_S + Q + [\mu+(s+1)\lambda]\ P_{s+1} = 0. \tag{4}$$

Solving recursively the system of Eqs. (3) and (4), we have

$$P_{s+2} = \frac{\mu+(s+1)\lambda}{\mu+(s+2)\lambda} P_{s+1}$$

$$P_{s+3} = \frac{\mu+(s+2)\lambda}{\mu+(s+3)\lambda} P_{s+2}$$

$$= \frac{\mu+(s+2)\lambda}{\mu+(s+3)\lambda} \cdot \frac{\mu+(s+1)\lambda}{\mu+(s+2)\lambda} P_{s+1}$$

$$= \frac{\mu+(s+1)\lambda}{\mu+(s+3)\lambda} P_{s+1}$$

In general, we find

$$P_n = \frac{\mu+(s+1)\lambda}{\mu+n\lambda} P_{s+1}, \qquad n = s+1, \ldots\ldots, s+Q. \tag{5}$$

The value of P_{s+1} is determined by recognizing that

$$\sum_{i=s+1}^{s+Q} P_i = 1 . \tag{6}$$

Thus, substituting Eq. (5) into Eq. (6),

$$P_{s+1} = \frac{1}{\sum_{i=s+1}^{s+Q} \frac{\mu + (s+1)\lambda}{\mu + i\lambda}} \tag{7}$$

Substituting (7) in (5), we obtain for the steady state distribution of the stock level.

$$P_n = \frac{1}{(\mu + n\lambda) \sum_{i=s+1}^{s+Q} \frac{1}{\mu + i\lambda}} , n = s+1, \ldots, s+Q. \tag{8}$$

Thus, the expected on hand inventory in the steady state is

$$E(H) = \sum_{j=s+1}^{s+Q} jP_j = \frac{\sum_{j=s+1}^{s+Q} \frac{j}{\mu + j\lambda}}{\sum_{i=s+1}^{s+Q} \frac{1}{\mu + i\lambda}} \tag{9}$$

Let V_n be the time, during which, the inventory level stays at n. Then, following Lemma can be proved.

[Lemma] The probability density function of $f_{v_n}(t)$ is given by

$$f_{v_n}(t) = (\mu + n\lambda)e - (\mu + n\lambda)^t , t \geqslant 0$$

(proof) Let Y be the time until a demand occurs. And let Z_1 be the time until with unit loss occurs. Then v_n takes on the value equal to the minimum of values actually taken on by Y, Z_1, Z_2, ..., Z_n, that is $v_n = \min \{Y, Z_1, Z_2, \ldots, Z_n\}$. Now note that for any $t \geqslant 0$,

$$\begin{aligned} P\{V_n > t\} &= P\{Y > t, Z_1 > t, Z_2 > t, \ldots, Z_n > t\} \\ &= P\{Y > t\} \; P\{Z_1 > t\}, \ldots, P\{Z_n > t\} \\ &= e^{-\mu t} e^{-\lambda t} \ldots\ldots e^{-\lambda t} \\ &= \exp\{-(\mu + n\lambda)t\} . \end{aligned}$$

Thus, V_n has an exponential distribution with parameter $\mu + n\lambda$. (Q. E. D)

If E(V) denotes the expected elapsed time between successive orders. Then, E (V) is given by

$$E(V) = E(\sum_{i=s+1}^{s+Q} V_i) = \sum_{i=s+1}^{s+Q} E(V_i)$$

$$= \sum_{i=s+1}^{s+Q} \frac{1}{\mu + i\lambda} \tag{10}$$

Let the procurement cost consists of fixed cost of K and a variable cost C per unit. The holding cost is h per unit per unit time.

Under steady-state conditions, the expected procurement cost per unit time is (K + CQ)/E(V), and the expected holding cost per unit time is hE(H). From Eqs. (9) and (10), the expected cost per unit time is given by

$$C(s, Q) = \frac{K + CQ}{E(V)} + hE(H)$$

$$= \frac{K + CQ + h \sum_{j=s+1}^{s+Q} \frac{j}{\mu + j\lambda}}{\sum_{i=s+1}^{s+Q} \frac{1}{\mu + i\lambda}} \tag{11}$$

3. OPTIMALITY CONDITION

The above expression is to be minimized with respect to s and Q. Since E(V) is decreasing in a and E(H) is increasing in s, C (s, Q) is increasing in s. Thus, it is clear that the optimum value of s is s* = 0. The optimum value of Q is obtained by minimizing the following function,

$$C(Q) = \frac{K + CQ + h \sum_{j=1}^{Q} \frac{j}{\mu + j\lambda}}{\sum_{i=1}^{Q} \frac{1}{\mu + i\lambda}} \tag{12}$$

Observing that Q is a positive integer, the optimum value of Q satisfies the following conditions;

$$C(Q^*) - C(Q^* - 1) \leqslant 0$$

$$C(Q^*) - C(Q^* + 1) \leqslant 0$$

After some manipulations, the above inequality may be written as

$$(\lambda Q^* + \mu) \sum_{i=1}^{Q^*} \frac{1}{\mu + i\lambda} = Q^* \leqslant \frac{\lambda K}{C\lambda + h} \leqslant (\lambda Q^* + \lambda + \mu) \sum_{i=1}^{Q^*} \frac{1}{\mu + i\lambda} - Q^*. \tag{13}$$

4. NUMERICAL EXAMPLE

As an illustration to the above developed model, consider a system with the following parameter values; K = \$ 100 per order, C = \$ 4 per unit, h = \$ 1 per unit/year, μ = 100, and λ = 5. Applying the inequality (13), we find that the optimal value of Q is Q^* =38. Thus, s^* = 0 and S^* = 38. From Eq. (12), C (Q^* = 38) = \$ 1,217.94. Table 1 shows the optimal value of Q with different value of λ.

As the λ increases, Q^* decreases but the minimum expected cost per unit time increases.

Table 1. Optimal value of Q with different value of λ.

λ	Q^*	C (Q^*)
1	70	\$ 751.35
5	38	\$ 1,217.94
10	30	\$ 1,642.37
20	24	\$ 2,344.97

5. CONCLUDING REMARKS

The earlier works on the perishable problems assumed that the life time of product is fixed and known with certainty.

However, in many circumstances, this assumption is not realistic. In this paper, we analyzed the problem, in which, the life time of the items is random. It is felt that the model respresents the real situation of many inventory systems.

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