〈硏究論文〉

Journal of the Korean Institute of Industrial Engineers
Vol. 11, No. 1, June 1985

Periodic Replacement Policies with Minimal Repair Cost Limit

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Abstract

Periodic replacement policies are proposed for a system whose repair cost, when it fails, can be estimated by inspection. The system is replaced when it reaches age T (Policy A), or when it fails for the first time after age T (Policy B). If it fails before reaching age T, the repair cost is estimated and minimal repair is then undertaken if the estimated cost is less than a predetermined limit L; otherwise, the system is replaced. The expected cost rate functions are obtained, their behaviors are examined, and ways of obtaining optimal T and L are explored.

I. Introduction

Since Barlow and Hunter (1960) introduced a periodic replacement model with minimal repair, many replacement policies with minimal repairs in a single unit system have been proposed; Muth (1977), Park (1979), Nakagawa (1981), Phelps (1981, 1983) and Nakagawa and Kowada (1983). In these policies, minimal repair cost is implicitly assumed to be constant. In many situations, however, it varies in a random fashion. To take into account the random nature of repair costs, repair cost limit policies have been proposed by several authors; Hastings (1969), Nakagawa and Osaki (1974), Nguyen

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and Murthy (1980), Kaio and Osaki (1981) and Park (1983).

In repair cost limit policies, replacement decisions depend only on the cost of single repair. However, further cost savings are possible if replacement decisions are also based on the history of the system. For detailed discussions see Cleroux et al. (1979) and Beichelt (1982).

We propose periodic replacement policies which consider both the cost of repair and the age of the system.

Basic assumptions

1) Minimal repair costs are i.i.d. random variables, observable through inspection.
2) Replacements and minimal repairs take only negligible time.
3) Harzard rate of the system is IFR and not disturbed by minimal repairs.
4) Planning horizon is infinite.

Notation

$F(t)$	:	c.d.f. of failure time
$r(t), R(t)$	:	harzard rate, cumulative harzard
c_0, c_1	:	replacement cost, inspection cost
Y_n	:	failure time of n^{th} failure
L	:	minimal repair cost limit
T	:	replacement period
$G(x)$	:	c.d.f. of repair cost
E_L	:	expected value of repair costs not exceeding L
$G(L)$	:	probability that the repair cost exceeds L
$N(t)$	:	number of repairs to time t
$M(t)$	:	mean residual life function
μ	:	expected cost of minimal repair

II. Policy A

When the system fails before age T, its repair cost is estimated by inspection. If the estimated cost does not exceed a cost limit L, the system undergoes a minimal repair. The system is replaced at the first major failure or at age T, whichever occurs first. Here by a major failure we mean the failure whose repair cost exceeds L.

A. Expected cost during a replacement cycle

The expected cost during a replacement cycle is first obtained. Let S_c be the expected cost during

a replacement cycle and S_n be the expected cost of minimal repairs performed until age T or $(n-1)^{th}$, failure, whichever occurs first.
Then,

$$S_c = c_0 + \sum_{n=1}^{\infty} S_n \cdot F_n$$

where

$$F_n = p_r\,[n^{th} \text{ failure is first major failure}]$$

$$= \bar{G}(L)\; G(L)^{n-1}$$

The expected value of the repair cost not exceeding L is

$$E_L = \int_0^L x\, dG(x)/G(L).$$

Hence,

$$S_n = [\, nc_1 + (n-1) E_L]\; p_r(Y_n \leq T) + \sum_{j=0}^{n-1} j[C_1 + E_L]\; p_r(N(T) = j).$$

From Nakagawa and Kowada (1983),

$$p_r\,(N(T) = n) = \exp\,[-R(T)]\, R(T)^n/n!$$

and

$$p_r(Y_n \leq T) = p_r\,(N(T) \geq n) = 1 - \sum_{j=0}^{n-1} [R(T)]^j \exp[-R(T)]/j!$$

Therefore,

$$S_C = c_0 + \sum_{n=1}^{\infty} [nc_1 + (n-1) E_L][1 - \sum_{j=0}^{n-1} R(T)^j e^{-R(T)}/j!]$$

$$+ \sum_{j=0}^{n-1} [E_L + c_1]\, j\, R(T)^j e^{-R(T)}/j!]\;\; \bar{G}(L)\;\; G(L)^{n-1}$$

$$= c_0 + [c_1 + (E_L + c_1) G(L)/\bar{G}(L)][1 - e^{-R(T)\bar{G}(L)}]$$

B. Expected duration of a replacement cycle

The conditional expected duration S_f of a replacement cycle given that n^{th} failure is the first major failure is given by

$$S_f = T\;\; P_T(Y_n > T) + \int_0^T t\, dp_r(Y_n \leq t) = \int_0^T P_r(Y_n > t)\, dt$$

$$= \int_0^T [\sum_{j=0}^{n-1} R(t)^j e^{-R(T)}/j!]\, dt.$$

Consequently, the expected duration S_d of a replacement cycle is given by

$$S_d = \sum_{n=1}^{\infty} S_f F_n = \sum_{n=1}^{\infty} [\int_0^T [\sum_{j=0}^{n-1} R(t)^j e^{-R(T)}/j!] dt] \overline{G}(L) G(L)^{n-1}$$
$$= \int_0^T e^{-R(T)\overline{G}(L)} dt . \quad (2)$$

C. Characteristics of policy A

From (1) and (2), the expected cost rate is obtained as

$$C(L, T) = S_c / S_d$$
$$= \{ c_0 + [c_1 + (c_1 + E_L) G(L)/\overline{G}(L)] [1 - e^{-R(T)\overline{G}(L)}] \}$$
$$[\int_0^T e^{-R(T)\overline{G}(L)} dt]^{-1} \quad (3)$$

Thus, to obtain the optimal policy, we seek the values of L and T which minimize C(L, T). The exact optimal values are difficult to obtain. Therefore the behavior of C(L, T) is examined, and ways of obtaining optimal L and T are explored.

i) When L goes to infinity and the inspection cost is negligible, the expected cost rate becomes

$$C_1(T) = \lim_{L \to \infty} C(L, T) |_{c_1=0} \quad (4)$$
$$= [c_0 + R(T)] / T$$

Note that (4) is the same as the one given by Barlow and Hunter (1960).

ii) When T goes to infinity and the inspection cost is negligible, the expected cost rate becomes

$$C_2(L) = \lim_{T \to \infty} C(L, T) |_{c_1=0}$$
$$= [c_0 + E_L G(L) / \overline{G}(L)] / [\int_0^{\infty} e^{-R(t)\overline{G}(L)} dt]$$

If we assume that the failure time follows a Weibull distribution and the repair cost an exponential distribution, the result is the same as that of Park (1983).

iii) The behavior of C(L, T) is now investigated for a given L. For simplicity of notations, let

$p = \overline{G}(L)$,

$a = c_1 + [E_L + c_1] (1 - p)/p$,

and

$$b = c_0 + a.$$

Then, the expected cost rate for a given L can be written as

$$C(T|L) = [b - ae^{-pR(T)}][\int_0^T e^{-pR(t)} dt]^{-1} \tag{5}$$

A necessary condition for T to minimize C(T|L) is obtained by setting the derivative of C(T|L) equal to 0, i.e.,

$$pr(T)\int_0^T e^{-pR(t)} dt + e^{-pR(T)} = b/a \tag{6}$$

If r(t) is strictly increasing, the left hand side of (6) is increasing in T and there exists a unique solution T* satisfying (6).

iv) Let L be also a decision variable and write $k=L/c_0$. Here parameter k is the repair cost limit expressed as a fraction of replacement cost c_0. In practical situations, we can assume that $0 < k < 1$. By varying k from 0 to 1, T^* for each given k can be obtained by the method discussed in iii), and the effect of parameter k to the expected cost rate can be examined. k* at which c(T*|k) is the smallest among the given values of k can be used.

D. A numerical example

A simple example is considered with –

$$F(t) = 1 - \exp(-t^{1.5}),\ G(x) = 1 - \exp(-x) \text{ and } c_1 = 0.$$

Under policy A, the local optimal solution is obtained by a searching method (Hooke and Jeeves algorithm) with T = T* and L = 1 as initial point.

Table 1. Comparision of the expected cost rates of various replacement policies (c_1=0, μ=1).

c_0	1.1	1.5	1.9	2.3	2.7
Barlow and Hunter	1.952	2.163	2.341	2.496	2.632
Park(1979)	1.219	1.662	1.927	2.143	2.325
Policy A	1.046	1.267	1.511	1.710	1.933

Table 1 gives minimum expected cost rates of three replacement policies for various replacement costs with fixed parameters (c_1=0, μ=1). As one might expect, policy A turns out to be superior to the other policies.

III. Policy B

Policy B differs from policy A in that the system is replaced at the first major failure or at the first failure after age T, whichever occurs first. The expected cost during a replacement cycle is the same as that of Policy A. Only the expected duration of a replacement cycle needs to be derived.

A. Expected duration of a replacement cycle

The conditional expected duration S_f of a replacement cycle given that n^{th} failure is the first major failure is given by

$$\begin{aligned} S_f &= [M(T)+T]P_r(Y_n > T) + \int_0^T t\, dP_r(Y_n \le t) \\ &= M(T)\, P_r(Y_n > T) + \int_0^T P_r(Y_n > t)\, dt \\ &= M(T) \sum_{j=0}^{n-1} R(T)^j e^{-R(T)} / j! + \int_0^T [\sum_{j=0}^{n-1} R(t)^j e^{-R(t)} / j!]\, dt. \end{aligned}$$

where $M(T) = \int_T^\infty [1-F(z)]\, dz / [1-F(T)]$ is the mean residual life function.

Consequently, the expected duration S_d of a replacement cycle is given by

$$\begin{aligned} S_d &= \sum_{n=1}^{\infty} S_f \cdot F_n \\ &= \sum_{n=1}^{\infty} \{ M(T) \sum_{j=0}^{n-1} R(T)^j e^{-R(T)} / j! + \int_0^T [\sum_{j=0}^{n-1} R(t)^j e^{-R(t)} / j!]\, dt \} \overline{G}(L)\, G(L)^{n-1} \\ &= M(T) e^{-R(T)\overline{G}(L)} + \int_0^T e^{-R(t)\overline{G}(L)}\, dt. \end{aligned} \tag{7}$$

B. Characteristics of policy B

From (1) and (7), the expected cost rate is obtained as

$$\begin{aligned} C(L,T) &= S_c / S_d \\ &= [c_0 + [c_1 + [c_2 + E_L]\, G(L) / \overline{G}(L)]\, [1 - e^{-R(T)\overline{G}(L)}]] \\ &\quad [M(T) e^{-R(T)\overline{G}(L)} + \int_0^T e^{-R(t)\overline{G}(L)}\, dt]^{-1} \end{aligned} \tag{8}$$

i) When L goes to infinity and the inspection cost is negligible, the expected cost rate becomes

$$\begin{aligned} C(T) &= \lim_{L\to\infty} C(L,T) \big|_{c_1=0} \\ &= [c_0 + R(T)] / [M(T)+T]. \end{aligned} \tag{9}$$

Note that (9) is the same as the one given by Muth (1977). When T goes to infinity and the inspection cost is negligible, the result is the same as that in policy A.

ii) The behavior of C(L, T) is now investigated for a given L. Using the same as notations in policy A,

$$C(T|L)=[b-ae^{-pR(T)}][M(T)e^{-pR(T)}+\int_0^T e^{-pR(t)}dt]^{-1}$$

A necessary condition for T to minimize C(T|L) is obtained by setting the derivative of C(T|L) equal to 0, i.e.,

$$apr(T)\,A(T)\,[M(T)A(T)+\int_0^T A(t)\,dt]$$

$$-[b-aA(T)][A(T)+M'(T)\,A(T)-pM(T)r(T)\,A(T)]=0$$

where $A(T) = e^{-pR(T)}$

Since $M'(T) = r(T)M(T)-1$, the left hand side of (10) becomes

$$Q(T) = A(T)r(T)\,[ap[M(T)A(T)+\int_0^T A(t)dt] - [b - aA(T)]\,[M(T)-M(T)p]]$$

$$= A(T)r(T)W(T).$$

where $W(T) = ap\int_0^T A(t)dt - bM(T) + bpM(T) + aA(T)M(T)$.

Now,

$$W'(T) = apA(T) - bM'(T) + bpM'(T) + a\,[-pr(T)A(T)]\,M(T) + aA(T)M'(T)$$

$$= apA(T)\,[1 - r(T)M(T)] + [-b + bp + aA(T)]\,M'(T)$$

$$= -(1-p)M'(T)\,[b-aA(T)].$$

Here $0 < p < 1$ and $b-aA(T) = S_f > 0$. Hence if r(T) is strictly increasing, then $M'(T) < 0$ (see Bryson and Siddiqui (1969)) and W(T) is strictly increasing in T. Furthermore, $W(0) < 0$, $A(T) > 0$ and $r(T) \geqslant 0$. Therefore a unique nonzero T satisfying (10) exists.

iv) If L is also a decision varaible, the method discussed in policy A can be used.

IV. Concluding remarks

Generalized replacement policies with minimal repair cost limit are proposed. The expected cost rate functions are obtained, those behaviors are examined, and ways of obtaining optimal T and L are explored. The number of minor failures can also be considered as a decision variable instead of system age, and a repair limit sequence policy proposed by Beichelt and Fischer (1980) may be applied to our policies.

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