〈研究論文〉

Journal of the Korean Institute of Industrial Engineers
Vol. 11, No. 2, Dec. 1985

Optimal Numbers of Repeat Inspections with Decreasing Detection Probability

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Abstract

Optimal numbers of repeat inspections are obtained for a single inspector who has a fixed probability of detecting a nonconforming item on each inspection and will continue to inspect until further inspection is not warranted when comparing the expected increase of total gain with the inspection cost. It is assumed that the detection probability decreases as the number of repeat inspections increases, and that the lot to be inspected contains an unknown but Poisson distributed number of nonconforming items.

1. Introduction

Inspection is used for preventing nonconforming items or materials from proceeding further in the processing or from being sold to a customer. Such a process necessarily entails errors of inspection, and it is not uncommon to find error figures of twenty-five percent or higher for even the most experienced inspection personnel (Jacobson, 1952). One common practice employed in industry to reduce the effects of human errors in the case of a critical component is to make reinspection, i.e., the use of 'inspection redundancy' (Swain, 1970), although this may also be less than perfect.

Raouf, et al. (1981) consider a mathematical model for determining the optimal number of inspection stages in a multi-characteristic inspection plan for a product that has several critical characteristics to be inspected.

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Yang, et al. (1982) give stopping rules for a proofreader who has a fixed probability p of detecting a misprint in proofsheets which contain an unknown but Poisson distributed number of misprints. Tierney (1983) points out that the conclusions can be extremely misleading if the detection probability varies with the types of misprints.

In this paper, the problem of optimally choosing the number of repeat inspections is considered for the case of decreasing detection probability.

Basic assumptions

1) The detection probability decreases with the number of repeat inspections.
2) The number of nonconforming items in the lot follows Poisson distribution.
3) Cost function is linear in both the number of repeat inspections and the number of nonconforming items.
4) At the end of the kth inspection, the expected number of nonconforming items detected at the (k+1)th inspection is greater than those of all inspections thereafter.

Notations

p_i detection probability at ith inspection

N_i number of nonconforming items detected at ith inspection

n_i observed value of N_i

s_k $\sum_{i=1}^{k} n_i$

μ_k $\sum_{i=1}^{k} i n_i$

c_1 inspection cost

c_2 penalty cost incurred by an undetected nonconforming item

$V(n_1,\ldots,n_k)$ expected gain at the end of the kth inspection

$E_t(N_x \mid n_1,\ldots,n_t)$ the posterior expectation of N_x given $n_1, \ldots, n_t$

M total number of nonconforming items in the lot

λ Poisson parameter of the distribution of M

ρ $\prod_{i=1}^{k} (1-p_i)$

ρ_1 $\prod_{i=1}^{k} (1-p/i)$

$$\rho_2 \qquad \prod_{i=1}^{k} (1-p^i)$$

2. Optimizing Procedures

The consturction of the optimal procedures is based on the principle of optimality. Just after the kth inspection, the inspector's gain is $c_2 s_k - c_1 k$. At this stage, the inspector either may stop and accept the gain or may, at a cost c_1, take another 100% inspection.

The inspector's expected gain, $V(n_1,...,n_k)$, when the optimal procedure is continued from the position $(n_1,...,n_k)$, is (see DeGroot, 1970) as follows.

$$(1) \quad V(n_1, ..., n_k) = \mathrm{Max}\ [c_2 s_k - c_1 k, E\{ V(n_1, ..., n_k, N_{k+1}) \}].$$

Equation (1) is, in general, difficult to solve. However, if Assumption 4 holds, i.e., for all $j > 0$ and all $k \geqslant 0$,

$$(2) \quad E_k \ldots E_{k+j-1} (N_{k+j} \mid n_1, ..., n_k, N_{k+1}, ..., N_{k+j-1}) \leqslant E_k(N_{k+1} \mid n_1, ..., n_k),$$

then it follows that the optimal procedure is just to look ahead only one stage from any given position. Therefore, if (2) holds, the optimal procedure is to stop inspection at stage k if and only if

$$c_2 s_k - c_1 k \geqslant c_2 [s_k + E_k (N_{k+1} \mid n_1, ..., n_k)] - c_1(k+1),$$

or, equivalently, if and only if

$$(3) \quad E_k (N_{k+1} \mid n_1, ..., n_k) \leqslant c_1 / c_2.$$

Inequality (2) is the consequence of the following lemma due to Yang, et al. (1982).

Lemma 1. If, for all $k \geqslant 0$

$$E_k E_{k+1} (N_{k+2} \mid n_1, ..., n_k, N_{k+1}) \leqslant E_k (N_{k+1} \mid n_1, ..., n_k),$$

then (2) holds.

3. Optimal Strategies When the Detection Probability Decreases

We first assume that p_i's and λ, are all known. Since the inspection of items corresponds to the Bernoulli trials, the joint distribution of $N_1, \ldots, N_k$ given M=m is, for each fixed k,

$$f(n_1, \ldots, n_k \mid m) = \frac{m!}{(m-s_k)! \prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} p_i^{n_i} (1-p_i)^{m-s_k},$$

where $s_k = \sum_{i=1}^{k} n_i$.

Since M is assumed to possess a Poisson distribution with parameter λ, the posterior distribution of M given $n_1, \ldots, n_k$ is

$$f(m \mid n_1, \ldots, n_k) = \frac{e^{-\lambda\rho} (\lambda\rho)^{m-s_k}}{(m-s_k)!}, \qquad m = s_k, s_k+1, \ldots,$$

where $\rho = \prod_{i=1}^{k} (1-p_i)$.

Moreover, the conditional distribution of N_{k+1} given $n_1, \ldots, n_k$, M can be shown to be binomial with parameters $M-s_k$ and p_{k+1}. Hence,

$$p_{k+1}(j) = Pr[N_{k+1}=j \mid n_1, \cdots, n_k]$$

$$= \sum_{m=s_k}^{\infty} Pr[N_{k+1}=j \mid m, n_1, \cdots, n_k] f(m \mid n_1, \cdots, n_k),$$

which is Poisson with parameter $\lambda\rho p_{k+1}$.

Thus,

(4) $\quad E_k(N_{k+1} \mid n_1, \cdots, n_k) = \lambda\rho p_{k+1}$.

Since the right hand side of (4) is independent of $n_1, \ldots, n_k$, the condition of Lemma 1 is obviously satisfied.

Therefore (3) implies that the optimal number of repeat inspections is, regardless of the outcomes $(n_1, n_2, \ldots, n_k)$, the smallest nonnegative interger k satisfying

(5) $\quad \lambda\rho p_{k+1} \leqq c_1/c_2$.

Note that the optimal procedure does not depend upon the actual values c_1 and c_2, but only on their ratio.

When the detection probabilities, p_i's, are known but Poisson parameter λ is unknown, the gamma conjugate prior distribution with parameters a and b is assumed for unknown parameter λ. That is,

$$f(\lambda) = b^a/\Gamma(a) \ \lambda^{a-1} e^{-b\lambda} \quad \lambda > 0.$$

Then, we obtain, after some simplifications,

$$f(m \mid n_1, \cdots, n_k) = \frac{\Gamma(m+a)}{\Gamma(m-s_k+1)\ \Gamma(s_k+a)} \rho^{m-s_k} (1+b)^{-m-a} (1+b-\rho)^{s_k+a}, \ m = s_k, s_k+1, \cdots$$

and $$p_{k+1}(j) = \frac{\Gamma(s_k+j+a)}{\Gamma(j+1)\ \Gamma(s_k+a)} (p_{k+1}\rho)^j [1+b-\rho]^{s_k+a} \ [1+b-(1-p_{k+1})\rho]^{-(s_k+j+a)}.$$

Hence

(6) $$E_k(N_{k+1} \mid n_1, \cdots, n_k) = p_{k+1}\rho(s_k+a)[1+b-\rho]^{-1}.$$

But $$E_k E_{k+1}(N_{k+2} \mid n_1, \cdots, n_k, N_{k+1})$$

$$= \sum_{j=0}^{\infty} (s_k+a+j)\ p_{k+2}(1-p_{k+1})\rho[1+b-(1-p_{k+1})\rho]^{-1} p_{k+1}(j)$$

$$= p_{k+2}(1-p_{k+1})\rho(s_k+a)[1+b-\rho]^{-1}$$

Since k is nonnegative integer and $1 > p_1 \geqslant \ldots \geqslant p_k \geqslant \ldots > 0$, the condition of Lemma 1 is satisfied. Thus, (3) implies that the optimal strategy is to inspect k times where k is the smallest nonnegative integer satisfying

(7) $$p_{k+1}\rho(s_k+a)[1+b-\rho]^{-1} \leqq c_1/c_2$$

Note that in this case the optimal strategy depends on s_k.

4. Numerical Examples

The procedure for obtaining optimal number of repeat inspections is illustrated by the following examples using the software failure data of Goel (1979). The data are originally from the U.S. Navy Fleet Computer Programming Center, and consist of the errors in the development of software for the real time, multicomputer complex which forms the core of the Naval Tactical Data System (NTDS). The NTDS software consists of some 38 different modules and the data used here are based on the trouble reports for one of the larger modules.

Example 1. If we modify the NTDS data by dividing it into fixed time intervals of equal length, 100 days, the inspector's probability of detecting an error in NTDS software is

$$p_k = .7843 \exp[-.579\,k], \quad k = 1, 2, \ldots$$

Suppose that the number of errors in the NTDS software follows Poisson distribution with parameter $\lambda = 40$, and that $c_1/c_2 = .25$. Then the optimal number of repeat inspections, k*, is the smallest integer satifying the inequality

$$(40)\ p_{k+1}\ \rho < .25$$

We can easily find k* = 6 and corresponding inspector's gain $c_2 s_k{}^* - c_1 k^* = 118$ units from Table 1.

Table 1. Sequential determination of k* in Example 1 ($c_1 = 1$)

k	p_k	$(40)\ p_{k+1}\ \rho$	s_k	$c_2 s_k - c_1 k$
1	.440	5.510	18	71
2	.246	2.331	23	90
3	.138	1.121	26	101
4	.077	.578	29	112
5	.043	.309	30	115
6*	.024	(.176 (≤.25)	31	118

Example 2. Suppose in Example 1 that parameter λ is not known and it is assumed to have gamma prior distribution with parameters a = 15 and b = 2.

Then k* is the smallest integer satisfying the inequality

$$\frac{p_{k+1}\,(s_k+15)\,\rho}{(1+2-\rho)} \leqq .25$$

Hence, we obtain k* = 4 and $c_2 s_k{}^* - c_1 k^* = 112$ units from Table 2.

Table 2. Sequential determination of k* in Example 2 ($c_1 = 1$)

k	s_k	$\rho(s_k+a)p_{k+1}/(1+b-\rho)$	$c_2 s_k - c_1 k$
1	18	1.864	71
2	23	.859	92
3	26	.436	104
4*	29	.238 (≤.25)	112

5. Two Special Cases

The general case of unknown detection probabilities $p_1, \ldots, p_k$ leads to complicated form of joint distribution which makes it difficult to derive optimal number of repeat inspections in meaningful form.

In this section, optimal numbers of repeat inspections are obtained for two special cases where the detection probability decreases (A) linearly with the reciprocal of the number of repeat inspections and (B) geometrically with the number of repeat inspections.

A. The Case of Reciprocally Decreasing Detection Probability

It is assumed that p has the conjugate beta prior distribution with parameters u and v, i.e.,

(8) $$f(p) = \frac{1}{B(u,v)} p^{u-1} (1-p)^{v-1}, \quad 0 < p < 1.$$

Then, with p/i as the detection probability at the ith inspection, the posterior distribution of M and p is given by

$$f(m, p \mid n_1, \cdots, n_k) = \frac{\Gamma(m+a)\, D_1^{-1} \rho_1^{m-s_k}}{\Gamma(m-s_k+1)\ \Gamma(s_k+a)\ (1+b)^{m+a}}$$
$$p^{u-1}(1-p)^{v-1} \prod_{i=1}^{k} [(p/i)^{n_i}(1-p/i)^{s_k-s_i}],$$

where $$D_1 = \int_0^1 [\prod_{i=1}^{k} (p/i)^{n_i}(1-p/i)^{s_k-s_i} p^{u-1}(1-p)^{v-1} [1+b-\rho_1]^{-(s_k+a)}]dp.$$

We also have

(9) $$E_k(N_{k+1} \mid n_1, \cdots, n_k)$$
$$= D_1^{-1}(s_k+a) \int_0^1 [\prod_{i=1}^{k} (p/i)^{n_i}(1-p/i)^{s_k-s_i}$$
$$p^{u-1}(1-p)^{v-1}(\frac{p}{k+1})\, \rho_1 [1+b-\rho_1]^{-(s_k+a+1)}]dp.$$

If we define $h_1(p)$ as

$$h_1(p) = \prod_{i=1}^{k} (p/i)^{n_i}(1-p/i)^{s_k-s_i}(\frac{p}{k+1})\rho_1 \quad p^{u-1}(1-p)^{v-1}[1+b-\rho_1]^{-(s_k+a+1)},$$

then $h_1(p)$ is continuous and positive for all $p \in (0, 1)$. Since k is a nonnegative integer and $0 < 1-p/(k+1) < 1$ for all $p \in (0, 1)$,

$$E_k(N_{k+1} \mid n_1, \cdots, n_k)/E_k E_{k+1}(N_{k+2} \mid n_1, \cdots, n_k, N_{k+1})$$
$$= \frac{\int_0^1 h_1(p)\ dp}{\frac{k+1}{k+2} \int_0^1 h_1(p)\{1-p/(k+1)\}\ dp} > 1.$$

Thus, the condition of Lemma 1 obtains.

B. The Case of Geometrically Decreasing Detection Probability

Let p have the beta conjugate prior with parameters u and v as in (8). Then the posterior distribution of M and p is

$$f(m, p \mid n_1, \cdots, n_k) = \frac{D_2^{-1} \Gamma(m+a)}{\Gamma(m-s_k+1) \Gamma(s_k+a)} p^{\mu_k+u-1} (1+b)^{-m-a} \cdot \prod_{i=1}^{k} (1-p^i)^{s_k-s_i} (1-p)^{v-1} \rho_2^{m-s_k}$$

where $\mu_k = \sum_{i=1}^{k} in_i$ and

$$D_2 = \int_0^1 [p^{\mu_k+u-1} (1-p)^{v-1} \prod_{i=1}^{k} (1-p^i)^{s_k-s_i} [1+b-\rho_2]^{-s_k+a}] dp.$$

We also have

$$(10) \quad E_k(N_{k+1} \mid n_1, \cdots, n_k) = D_2^{-1} (s_k+a) \int_0^1 [p^{\mu_k+k+u} \rho_2 \prod_{i=1}^{k} (1-p^i)^{s_k-s_i} (1-p)^{v-1} [1+b-\rho_2]^{-(s_k+a)}] dp.$$

If we let

$$h_2(p) = p^{\mu_k+k+u} \rho_2 (1-p)^{v-1} \prod_{i=1}^{k} (1-p_i)^{s_k-s_i} [1+b-\rho_2]^{-(s_k+a+1)}$$

then $h_2(p)$ is also continuous and positive for all $p \in (0, 1)$. Since k is a nonnegative integer and $0 < (1-p^{k+1}) < 1$ for all $p \in (0, 1)$,

$$E_k(N_{k+1} \mid n_1, \cdots, n_k) / E_k E_{k+1} (N_{k+2} \mid n_1, \cdots, n_k, N_{k+1}) = \frac{\int_0^1 h_2(p)\, dp}{\int_0^1 h_2(p)\, p(1-p^{k+1})\, dp} > 1.$$

Therefore, the condition of Lemma 1 is satisfied.

6. Concluding Remarks

Optimal strategies for choosing the number of repeat inspections are obtained for a single inspector in the cases where the probability of detecting a nonconforming item has a general decreasing mode and in some special cases. When the detection probabilities are known the proposed strategies can be easily obtained, but when both the detection probabilities and Poisson parameter are unknown it becomes complicated and some approximations may be necessary for practical use. Except for the case of known p_i's and λ, the expected values of N_{k+1} given $n_1, \ldots, n_k$ such as (6), (9), and (10) are dependent on the observed values $(n_1, \ldots, n_k)$. Therefore, decisions should be made sequentially at each stage.

Although it is assumed here that all nonconforming items are equally easy or hard to detect, the approach can also be applied to the problem of finding optimal number of repeat inspections when there are different types of nonconformities.

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