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A GENERALIZATION OF BOOLEAN RINGS

By Hisao Tominaga and Adil Yaqub

Throughout the present note, R will represent an alternative ring (all of whose subrings generated by two elements are associative) with identity 1, C the set of elements c in R such that $cx=xc$ for all $x\in R$, and N the set of nilpotent elements in R.

Recently, A Melter [3, p. 220] considered the following condition which arose, presummably, in connection with logic:

(*) $(1-x+xy)(1-y+yx)=1$ if and only if $x=y$ $(x, y\in R)$,

and raised the following question: For which associative rings with identity does the condition (*) hold? T.M.K. Davison [1] gave the following characterization of associative rings satisfying (*): An associative ring R with 1 satisfies (*) if and only if (i) $x^5=x^3$ for all $x\in R$, (ii) $4x^2=4x$ for all $x\in R$, and (iii) for any idempotent $e\in R$ and any non-zero $x\in R$, $x\neq[x, e]$ $(=xe-ex)$. The present authors and Y. Hirano [2] proved recently that if an associative ring R with 1 satisfies (*) then R is commutative and R/N is a Boolean ring. Incidentally, that (*) implies commutativity is not apparent from Davison's characterization.

The purpose of this note is to give the following theorem with an elementary proof (cf. also the main theorem of [4]):

THEOREM. *Let R be an alternative ring with identity* 1, *and N the set of nilpotent elements in R. Then R satisfies* (*) *if and only if* (*a*) *R is commutative and R/N is a Boolean* (*associative*) *ring, and* (*b*) *$u^2=1$ for every unit u in R.*

PROOF. "Only if": Put $x=2=y$ in (*) to get $8=0$. Put $y=x$ in (*) to get $(1-x+x^2)^2=1$ and hence

$$(1)\qquad 2(x-x^2)=(x-x^2)^2.$$

Obviously, $(x-x^2)^6=8(x-x^2)^3=0$, and therefore

$$(2)\qquad x-x^2\in N.$$

By (1), $4x=2(x-x^2)-2(-x-(-x)^2)=(x-x^2)-(-x-(-x)^2)^2=-4x^3$, and hence

$$(3)\qquad 4x=4x^3.$$

Replacing x by $x+1$ in (3) and noting that $8=0$, we get $4(x+1)=4(x+1)^3=$

$4x^3+4x^2+4x+4$, and therefore by (3),

(4) $$4x=4x^2.$$

Now, since $x^4=2x^3-3x^2+2x$ by (1), we see that $x^5=2(2x^3-3x^2+2x)-3x^3+2x^2=x^3-4x^2+4x$, and so by (4)

(5) $$x^5=x^3.$$

In particular, $u^2=1$ (or equivalently $u^{-1}=u$) for every unit u in R, proving (b). Let u, v be units in R. Then, $uv=(vu)^{-1}=vu$; in particular, *if* $a\in N$, $b\in N$, then $[a, b]=[1+a, 1+b]=0$. Now, let $e\in R$, $e^2=e$, and let $t\in R$. Set $a=et(1-e)$. Then $a^2=0=ae$ and $ea=a$. Since

$$\{1-(a+e)+(a+e)e\}\{1-e+e(a+e)\}=(1-a)(1+a)=1,$$

(*) gives $a+e=e$, and thus $a=0$, i.e., $et=ete$. A similar argument, putting $a=(1-e)te$, shows that $te=ete$, and hence $et=te$. Thus every idempotent element in R belongs to C. By (5), $x^8=x^6=x^4$, and hence x^4 is in C. Now, let $a\in N$, $x\in R$. Since $x-x^2$ and x^2-x^4 are both in N by (2) and x^4 is in C, we have $[a, x]=[a, x-x^4]=[a, x-x^2]+[a, x^2-x^4]=0$, and hence N is an ideal contained in C. Therefore $x=(x-x^2)+(x^2-x^4)+x^4\in C$ for all x in R, that is R is commutative. Now, let x, y, $z\in R$. Then, by (2) and the commutativity of R,

$$\begin{aligned} x(yz)-(xy)z &= x(zy)+z(xy)-2z(xy) \\ &= (x+z)\{(x+z)y\}-(x+z)^2y+ \\ &\quad \{(x+z)^2-(x+z)\}y-(x^2-x)y-(z^2-z)y-2z(xy)\in N, \end{aligned}$$

which proves that R/N is associative, and therefore Boolean, proving (a).

"If": Since R/N is a Boolean ring, $x-x^2\in N$ and hence $1-x+x^2$ is a unit in R. Therefore, by (b), $(1-x+x^2)^2=1$, which proves one part of (*). To prove the other part of (*), suppose that $(1-x+xy)\ (1-y+yx)=1$. Then, by (b) and the commutativity of R,

$$0=(1-x+xy)\ \{1-(1-x+xy)\ (1-y+yx)\}=1-x+xy-(1-y+yx)=y-x,$$

i.e., $x=y$. This proves the "if" part, and the theorem is proved.

Okayama University
Okayama 700
Japan

University of California
Santa Barbara
California 93106
U.S.A.

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