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SEMI-NORMAL SPACES

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1. Introduction

Semi open sets were introduced in 1963[10]. Since 1963 semi open sets have been used to define and investigate many new topological properties. In [11], [13], [14], and [7], semi-T_i, $i=0,1,2$, s-regular, s-normal, and semi compact were defined, respectively, by replacing the word open in the definition of T_i, $i=0,1,2$, regular, normal, and compact by semi open, respectively, and it was shown that semi-T_i, $i=0,1,2$, s-regular, and s-normal are weaker than T_i, $i=0,1,2$, regular, and normal, respectively, and that compactness is weaker than semi compactness. In this paper semi open sets are used to define and investigate a new topological property. Listed below are other definitions and results that will be utilized in this paper.

DEFINITION 1.1. If (X, T) is a space and $A \subset X$, then A is semi open, denoted by $A \in SO(X, T)$, iff there exists $O \in T$ such that $O \subset A \subset \bar{O}$ [10].

DEFINITION 1.2. Let (X, T) be a space and let A, $B \subset X$. Then A is semi closed iff $X-A$ is semi open and the semi closure of B, denoted by $s\mathrm{cl}B$, is the intersection of all semi closed sets containing B [1].

DEFINITION 1.3. Let R be the equivalence relation on a space (X, T) defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the T_0-identification space of (X, T) is $(X_0, Q(X_0))$, where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0, which is T_0. The natural map $P_0 : (X, T) \rightarrow (X_0, Q(X_0))$ is continuous, closed, open, and $P_0^{-1}(P_0(U)) = U$ for all $U \in T$ [4].

DEFINITION 1.4. A space (X, T) is R_0 iff one of the following equivalent conditions is satisfied: (a) for each $O \in T$ and $x \in O$, $\overline{\{x\}} \subset O$, (b) $X_0 = \{\overline{\{x\}} \mid x \in X\}$, and (c) $(X_0, Q(X_0))$ is T_1 [4].

DEFINITION 1.5. A space (X, T) is semi-R_0 iff for each $O \in SO(X, T)$ and $x \in O$, $s\mathrm{cl}\{x\} \subset O$ [12].

DEFINITION 1.6. A space (X, T) is R_1 iff for each pair x, $y \in X$ such that

$\overline{\{x\}} \neq \overline{\{y\}}$, there exists disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [3].

DEFINITION 1.7. A space (X, T) is s-weakly Hausdorff iff one of the following equivalent conditions is satisfied: (a) for each pair $x, y \in X$ such that $\{\bar{x}\} \neq \{\bar{y}\}$, there exist disjoint semi open sets U and V such that $x \in U$ and $y \in V$ and (b) $(X_0, Q(X_0))$ is semi-T_2 [8].

DEFINITION 1.8. A space (X, T) is semi-R_1 iff for each pair $x, y \in X$ such that $s\mathrm{cl}\{x\} \neq s\mathrm{cl}\{y\}$, there exist disjoint semi open sets U and V such that $s\mathrm{cl}\{x\} \subset U$ and $s\mathrm{cl}\{y\} \subset V$ [5].

DEFINITION 1.9. Let R be the equivalence relation on a space (X, T) defined by xRy iff $s\mathrm{cl}\{x\} = s\mathrm{cl}\{y\}$. Then the semi-$T_0$-identification space of (X, T) is $(X_s, Q(X_s))$, where X_s is the set of equivalence classes of R and $Q(X_s)$ is the decomposition topology on X_s. The natural map $P_s : (X, T) \rightarrow (X_s, Q(X_s))$ is continuous, closed, open, and $P_s^{-1}P_s(U) = U$ for all $U \in SO(X, T)$ [6].

DEFINITION 1.10. A subset A of a space (X, T) is an α-set iff $A \subset \mathrm{Int}(\overline{\mathrm{Int}(A)})$ [15].

DEFINITION 1.11. A 1-1 function from one space onto another space is a semihomeomorphism iff images of semi open sets are semi open and inverses of semi open sets are semi open. A property of topological spaces preserved by semihomeomorphisms is called a semi topological property [2].

DEFINITION 1.12. A space (X, T) is semi-regular iff for each semi closed set A and $x \notin A$, there exist disjoint semi open sets U and V such that $x \in U$ and $A \subset V$ [9].

DEFINITION 1.13. A space (X, T) is semi-normal iff for each pair of disjoint semi closed sets A and B, there exist disjoint semi open sets U and V such that $A \subset U$ and $B \subset V$.

2. Characterizations and relationships with other separation axioms

THEOREM 2.1. *The following are equivalent*:
(a) (X, T) *is semi-normal.*

(*b*) *every semihomeomorphic image is semi-normal.*

(*c*) *for each semi closed set A and each semi open set U such that $A \subset U$, there exists a semi open set V such that $A \subset V \subset \mathrm{scl}\, V \subset U$.*

(*d*) *for each semi closed set A and each semi open set U such that $A \subset U$, there exists a semi open, semi closed set V such that $A \subset V \subset U$.*

(*e*) *for each α-set A, (A, T_A) is semi-normal.*

(*f*) *$(X_s, Q(X_s))$ is semi-normal.*

PROOF. The straightforward proof that (a) and (b) are equivalent and (a) implies (c) is omitted. Also, since the semi closure of each semi open set is semi open, then (d) follows immediately from (c).

(d) implies (e): Let A be an α-set. Let B and C be disjoint semi closed sets in (A, T_A). Let $U = A - C \in SO(A, T_A)$. Since $U \in SO(A, T_A)$ and $A \in SO(X, T)$, then $U \in SO(X, T)$ [11] and there exists $W \in T$ such that $W \subset U \subset \overline{W}$. Then $W \subset U \cup \mathrm{scl}_X B \subset \overline{W}$, which implies $U \cup \mathrm{scl}_X B$ is semi open in X containing $\mathrm{scl}_X B$, and there exists a semi open, semi closed set V such that $\mathrm{scl}_X B \subset V \subset U \cup \mathrm{scl}_X B$. Since V is semi open, semi closed in X and A is an α-set, then $A \cap V$ and $A \cap (X - V)$ are semi open in A [15], which implies $A \cap V$ is semi open, semi closed in A. Let $x \in A \cap \mathrm{scl}_X B$. Let $Y \in SO(A, T_A)$ such that $x \in Y$. Then $Y \in SO(X, T)$ and $Y \cap B \neq \phi$, which implies $x \in \mathrm{scl}_A B = B$. Thus $\mathrm{scl}_X B = B \cup ((\mathrm{scl}_X B) - A)$ and $B \subset V \cap A \subset (U \cup \mathrm{scl}_X B) \cap A = (U \cup ((\mathrm{scl}_X B) - A)) \cap A = U \cap A = U$. Then $B \subset V \in SO(A, T_A)$ and $C \subset A - V \in SO(A, T_A)$.

(e) implies (f): Since X is an α-set, then (X, T) is semi-normal. Let $\mathscr{A}$ and $\mathscr{B}$ be disjoint semi closed sets in X_s. Since P_s is continuous and open, then $P_s^{-1}(\mathscr{A})$ and $P_s^{-1}(\mathscr{B})$ are disjoint semi closed sets in X [16]. Then $P_s^{-1}(\mathscr{A}) \subset X - P_s^{-1}(\mathscr{B}) \in SO(X, T)$ and by the argument above, there exists a semi open, semi closed set V such that $P_s^{-1}(\mathscr{A}) \subset V \subset X - P_s^{-1}(\mathscr{B})$. Since P_s is continuous, open, and $P_s^{-1}(P_s(U)) = U$ for all $U \in SO(X, T)$, then $P_s(V)$ is semi open, semi closed in X_s [10], $\mathscr{A} \subset P_s(V)$, and $\mathscr{B} \subset X_s - P_s(V)$.

(f) implies (a): Let A and B be disjoint semi closed sets in X. Since P_s is continuous, open, and $P_s^{-1}(P_s(C)) = C$ for each semi closed set C in X, then $P_s(A)$ and $P_s(B)$ are disjoint semi closed sets in X_s and there exist disjoint semi open sets $\mathscr{U}$ and $\mathscr{V}$ in X_s containing $P_s(A)$ and $P_s(B)$, respectively. Since

P_s is continuous and open, then $P_s^{-1}(\mathscr{U})$ and $P_s^{-1}(\mathscr{V})$ are disjoint semi open sets in X containing A and B, respectively.

By (b) above semi-normal is a semi topological property. Also, since every open set is an α-set, then α-set in (e) could be replaced by open. The following example shows that α-set in (e) can not be replaced by semi open, closed.

EXAMPLE 2.1. Let $X=\{a, b, c, d\}$, let $T=\{\{a\}, \{b\}, \{a, b\}, X, \phi\}$, and let $Y=\{a, c, d\}$. Then (X, T) is semi-normal, Y is a semi open, closed set in X, and (Y, T_Y) is not semi-normal.

THEOREM 2.2. *If (X, T) is semi-normal, then (X, T) is s-normal.*

The straightforward proof is omitted.

The space (Y, T_Y) in Example 2.1 shows that converse of Theorem 2.2 is false even when the space is semi compact and normal.

THEOREM 2.3. *If (X,T) is semi-normal, semi-R_0, then (X, T) is semi-regular.*

PROOF. Let A be semi closed and let $x\notin A$. Then $x\in X-A\in SO(X, T)$, which implies $\text{scl}\{x\}\subset X-A$, and there exist disjoint semi open sets U and V such that $x\in\text{scl}\{x\}\subset U$ and $A\subset V$.

The next result follows immediately from Theorem 2.3 and the fact that semi-regular implies semi-R_1 [9].

COROLLARY 2.1. *Every semi-normal, semi-R_0 space is semi-R_1.*

In [9] it was shown that for a semi compact space (X, T), the following are equivalent: (a) (X, T) is semi-regular, (b) (X, T) is s-regular, (c) (X, T) is s-weakly Hausdorff, (d) (X, T) is semi-R_0 and X_0 is finite, (e) (X, T) is semi-R_0 and $\{\text{scl}\{x\} \mid x\in X\}$ is finite, and (f) (X, T) is semi-R_1. Example 3.2 in [9], which is semi-regular, and semi-T_2 but not s-normal, shows that the converse of Theorem 2.3 and Corollary 2.1 is false even if the space is semi compact. Example 3.1 in [9], which is T_2 and not semi-regular, shows that T_2 does not imply semi-normal. Example 3.2 in [8], which is normal, regular, R_1, and semi-T_1 but not semi-R_1, shows that regular, R_1, and semi-T_1 does not imply semi-normal. If $X=\{a, b\}$ and $T=\{X, \phi, \{a\}\}$, then (X, T) is semi-normal and not semi-regular, regular, or semi-R_0. If S is the indiscrete topology on $X=\{a, b\}$, then (X, S) is semi-normal and not semi-T_0. Combining the spaces (X, T) and (X, S) with Example 3.1 and Example 3.2 in [9] and, Example 3.2 in [8] shows that semi-normal is independent with each of regular semi-regular, s-weakly Hausdroff, T_i and semi-T_i, $i=0, 1, 2$, and R_i and semi-R_i, $i=0, 1$. Combining the space (Y, T_y) in Example 2.1 with Example 1 in [14], which is semi compact, semi-normal, and not normal, shows that normal and semi-normal are independent even for semi compact spaces.

THEOREM 2.4. *If (X, T) is semi compact R_0, then the following are equivalent:*

(*a*) *(X, T) is semi-normal.*

(*b*) *(X, T) is s-normal.*

(*c*) *(X, T) is s-regular.*

(*d*) *(X, T) is s-weakly Hausdorff.*

(*e*) *(X, T) is semi-R_1.*

(*f*) *(X, T) is R_1.*

(*g*) *(X, T) is regular.*

(*h*) *(X, T) is completely regular.*

(*i*) *(X, T) is normal.*

(*j*) *(X, T) is semi-regular.*

(*k*) *$\{\mathrm{scl}\{x\} \mid x \in X\}$ is finite.*

(*l*) *$\{\overline{\{x\}} \mid x \in X\}$ is finite.*

PROOF. By Theorem 2.2, (a) implies (b) and by Theorem 3.3 in [7], (b) through (i) are equivalent. By Theorem 3.6 in [9], (c) and (j) are equivalent, and (j) implies (k). Since (X, T) is R_0, then $X_0 = \{\overline{\{x\}} \mid x \in X\}$ and (X, T) is semi-R_0, and by Theorem 3.6 in [9], (k) implies $X_0 = \{\overline{\{x\}} \mid x \in X\}$ is finite.

(l) implies (a): Since (X, T) is R_0, then $X_0 = \{\overline{\{x\}} \mid x \in X\}$ is a finite decomposition of X. Then $X_0 = \{\overline{\{x_i\}} \mid i \in \{1, \cdots, n\}\}$, where n is a natural number and $\overline{\{x_i\}} \cap \overline{\{x_j\}} \neq \phi$ iff $i=j$. If $O \in T$, then since (X, T) is R_0, $O = \bigcup_{x_j \in O} \overline{\{x_j\}}$, which is closed, which implies every open set is also closed. If $U \in SO(X, T)$, then there exists $W \in T$ such that $W \subset U \subset \overline{W} = W$, which implies every semi open set is open, closed. If A is semi closed and $U \in SO(X, T)$ containing A, then U is semi open, semi closed containing A and contained in U.

3. Images of semi-normal and semi-regular spaces

The following example shows that the continuous, closed, open image of a semi-normal, semi-regular space need not be semi-normal or semi-regular.

EXAMPLE 3.1. Let $X=\{a, b, c, d\}$, $T=\{\{a\}, \{d\}, \{a, d\}, \phi, X\}$, $Y=\{e, f, g\}$, $S=\{\{e\}, \phi, Y\}$, and $f=\{(a, e), (d, e), (b, f), (c, g)\}$. Then $f:(X, T)\rightarrow(Y, S)$ is continuous, closed, open, onto, (X, T) is semi-normal, semi-regular and (Y, S) is neither semi-normal nor semi-regular.

LEMMA 3.1. *If C is semi closed in X, then $P_0(C)$ is semi closed in X_0.*

PROOF. Since C is semi closed in X, then $X-C$ is semi open and there exists $Y\in T$ such that $Y\subset X-C\subset \overline{Y}$. Let $C_z\notin P_0(C)$, where C_z denotes the equivalence class of R containing z. Then $z\notin C$, $Y\cup\{z\}$ is semi open in X, $C_z\in\mathscr{W}=P_0(Y\cup\{z\})=P_0(Y)\cup\{C_z\}$, which is semi open, and since $P_0^{-1}(P_0(Y))=Y$, then $\mathscr{W}\cap P_0(C)=\phi$. Therefore $X_0-P_0(C)$ is semi open, which implies $P_0(C)$ is semi closed.

THEOREM 3.1. *If (X, T) is semi-regular, then so is $(X_0, Q(X_0))$ and if (X, T) is semi-normal, then so is $(X_0, Q(X_0))$.*

PROOF. Suppose (X, T) is semi-regular. Let $C_x\in X_0$ and let $\mathscr{U}\in SO(X_0, Q(X_0))$ such that $C_x\in\mathscr{U}$. Then $x\in P_0^{-1}(\mathscr{U})\in SO(X, T)$ and there exists a semi open, semi closed set V in X such that $x\in V\subset P_0^{-1}(\mathscr{U})$ (9). Then $C_x\in P_0(V)\subset\mathscr{U}$, where $P_0(V)$ is semi open, semi closed in X_0, which implies $(X_0, Q(X_0))$ is semi-regular [9]. The proof for semi-normal is similar to that for semi-regular above and is omitted.

The following example shows that the converse of Theorem 3.1 is false.

EXAMPLE 3.2. Let N denote the natural numbers, let T be the discrete topology on N, let e be the embedding map of (N, T) onto $\Pi\{I_f|f\in C^*(N, T)\}$, and let $(\beta(N), W)=(\overline{e(N)}, e)$ denote the Stone-Čech compactification of (N, T). Then $(\beta(N), W)$ is extremely disconnected, $e(N)$ is open in $\beta(N)$ and $\beta(N)$ has non isolated points [17]. Let x be a non isolated point of $\beta(N)$, let $X=e(N)\cup\{x\}$, and let T be the relative topology on X. Let $y\notin X$, let $Y=X\cup\{y\}$, and let $S=\{O\in T|x\notin O\}\cup\{O\cup\{y\}|x\in O\in T\}$. Then (Y, S) is not semi-regular or semi-normal since $scl\{x\}=\{x\}\neq\{y\}=scl\{y\}$ and there does not exist disjoint semi open sets containing x and y, respectively. Since (X, T) is semi-normal and semi-regular, (X, T) is homeomorphic to $(Y_0, Q(Y_0))$, and semi-normal and semi-regular are semi topological properties, then $(Y_0, Q(Y_0))$ is semi-normal and semi-regular.

THEOREM 3.2. *If (X, T) is semi compact, then the following are equivalent:*
(a) (X, T) is semi-regular.
(b) $(X_0, Q(X_0))$ is semi-regular.

PROOF. By Theorem 3.1, (a) implies (b).

(b) implies (a): Since $(X_0, Q(X_0))$ is semi-regular and T_0, then $(X_0, Q(X_0))$ is semi-R_1 and semi-T_0, which implies $(X_0, Q(X_0))$ is semi-T_2 [5]. Then (X, T) is semi compact and s-weakly Hausdorff, which implies (X, T) is semi-regular.

The space (Y, S) in Example 3.1 shows that semi-regular in Theorem 3.2 can not be replaced by semi-normal.

THEOREM 3.3. *If (X, T) is semi compact and R_0, then the following are equivalent:*
(a) (X, T) is semi-normal.
(b) $(X_0, Q(X_0))$ is semi-normal.

PROOF. By Theorem 3.1, (a) implies (b).

(b) implies (a): Since (X, T) is R_0, then $(X_0, Q(X_0))$ is T_1. Then $(X_0, Q(X_0))$ is semi-normal and semi-R_0, which implies $(X_0, Q(X_0))$ is semi-regular, and by the argument above $(X_0, Q(X_0))$ is semi-T_2. Then (X, T) is semi compact, R_0, and s-weakly Hausdorff, which implies (X, T) is semi-normal.

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