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BERGMAN SPACES ON BOUNDED SYMMETRIC DOMAINS

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1. Introduction

Let D be a bounded symmetric domain in $\boldsymbol{C}^n (n>1)$, and $0\in D$. D is circular and star-shaped with respect to the origin, i.e. $tz\in D$ when $z\in D$ and $t\in \boldsymbol{C}$ with $|t|\leq 1$ [3]. Let Γ be the group of holomorphic automorphisms of D ; Γ is transitive on D [3]. D has a Bergman-Šilov boundary b.

We denote by $H(D)$ the space of holomorphic functions on D.

For $p>0$ the Bergman space A^p is defined on D by

$$A^p=A^p(D)=\left\{f: f\in H(D) \text{ and } ||f||_{A^p}=\left(\frac{1}{V}\int_D |f(z)|^p dV_z\right)^{1/p}<\infty\right\},$$

or equivalently [4]

$$A^p=\left\{f: f\in H(D) \text{ and } ||f||_{A^p}=\sup_{0\leq r<1}\left(\frac{1}{V}\int_D |f(rz)|^p dV_z\right)^{1/p}<\infty\right\}$$

where V is the euclidean volume of D and dV_z is the euclidean volume element at $z\in D$.

The Hardy space $H^p (p>0)$ is defined on D by

$$H^p=H^p(D)=\left\{f: f\in H(D) \text{ and } ||f||_{H^p}=\sup_{0\leq r<1}\left(\frac{1}{V'}\int_b |f(rt)|^p ds_t\right)^{1/p}<\infty\right\}$$

where V' is the euclidean volume of b and ds_t is the euclidean volume element at $t\in b$.

In Section 2 we study Banach space properties of A^p spaces. In Section 3 we prove the inclusion relation between H^p and A^p and a necessary and sufficient condition for a holomorphic funtion to be in A^p is given. Finally, we generalize a result in H^p spaces to A^p spaces.

2. Banach space properties of A^p space

We have the following theorem:

THEOREM 1. *Let D be a bounded symmetric domain. Then A^p is a Banach space for $1\leq p<\infty$, and a complete linear Hausdorff space for $0<p<1$.*

For the proof of Theorem 1, we need the following lemma:

LEMMA. *Let D be a bounded symmetric domain and $f\in A^p$ $(0<p<\infty)$. Then for $z\in D$, there exists a constant $C(z, D, p)$ such that*

$$|f(z)|\leq C(z, D, p)\|f\|_{A^p} \tag{2.1}$$

For any compact set G of D, there exists a constant $C(G, D, p)$ such that for $z\in G$

$$|f(z)|\leq C(G, D, p)\|f\|_{A^p}.$$

Hence if $\|f-f_n\|_{A^p}\to 0$ as $n\to\infty$, then $f_n\to f$ uniformly on compact subsets of D.

PROOF. For fixed r, $0<r<1$, $|f(rz)|^p$ is plurisubharmonic on $\bar{D}$ and $|f_r(z)|^p\in L^1(D)$. Then by corollary [6, p. 193]

$$|f_r(z)|^p\leq\int_D\frac{|K(z,\bar{\xi})|^2}{K(z,\bar{z})}\,|f(r\xi)|^p dV_\xi$$

where $K(z,\bar{\xi})$ is the Bergman kernel of D.
Hence

$$|f_r(z)|^p\leq V\int_D|K(z,\bar{\xi})|^2|f(r\xi)|^p dV_\xi,$$

since $K(z,\bar{z})\geq\frac{1}{V}$. For fixed $z\in D$, $K(z,\bar{\xi})$ is continuous with respect to $\bar{\xi}$ on $\bar{D}$. Therefore

$$|f_r(z)|^p\leq C(z, D)\int_D|f(r\xi)|^p dV_\xi.$$

Letting $r\to 1$, we get

$$|f(z)|\leq C(z, D, p)\ \|f\|_{A^p}.$$

which proves (2.1).

The proof of Theorem 1, is analoguous to the corresponding result for H^p spaces [1, Theorem 5].

3. Inclusion relation between H^p and A^p.

We have the following theorem:

THEOREM 2. *Let D be a bounded symmetric domain. Then for $p>0$, $H^p(D)\subset A^p(D)$ and $\|f\|_{A^p}\leq\|f\|_{H^p}$.*

PROOF. Let $f\in H^p$. By [1, Theorem 3]

$$|f(z)|^p\leq\int_b P(z,t)\phi(t)ds_t \tag{3.1}$$

where $\phi\in L^1(b)$. Integrate both sides of (3.1) over D,

$$\frac{1}{V}\int_D|f(z)|^p dV_z\leq\frac{1}{V}\int_D\Big(\int_b P(z,t)\phi(t)ds_t\Big)dV_z. \tag{3.2}$$

By Fubini's theorem used on the right side of (3.2) and [6, Corollary 1]

$$\frac{1}{V}\int_D |f(z)|^p dV_z \le \frac{1}{V}\int_b \phi(t)ds_t = ||f||^p_{H^p}$$

[1, Remark, p. 525]. Hence $f\in A^p$ and $||f||_{A^p}\le ||f||_{H^p}$. H^p is a proper subset of A^p, for $N=1$ see [5, Theorem 1].

A necessary and sufficient condition for a holomorphic function to belong to the space $A^p(p>0)$ is as follows:

THEOREM 3. *Let D be a bounded symmetric domain, $z_0\in D_r$, $0<r<1$, and f holomorphic on D. Then $f\in A^p$ if and only if there exists a constant $C(z_0)$, independent of r, such that*

$$\int_D T(z_0, \bar{\xi})|f_r(\xi)|^p dV_\xi \le C(z_0) \tag{3.3}$$

where

$$T(z, \bar{\xi}) = \frac{|K(z, \bar{\xi})|^2}{K(z, \bar{z})}$$

and $K(z, \bar{\xi})$ is the Bergman kernel of D.

PROOF. Assume that $f\in A^p$. For fixed $z\in D$, $\dfrac{K(z, \bar{\xi})}{K(z, z)}$ is continuous with respect to $\bar{\xi}$ on $\bar{D}$. Therefore

$$\int_D T(z_0, \bar{\xi})|f(r\xi)|^p dV_\xi \le \max_{\xi\in\bar{D}} T(z_0, \bar{\xi})\int_D |f(rz)|^p dV_z$$
$$\le C(z_0)||f||_{A^p} = C(z_0).$$

This proves the necessity of (3.3).

Conversely, assume (3.3) is satisfied. Then

$$\frac{1}{V}\int_D |f(r\xi')|^p dV_{\xi'} = \int_D T(0, \xi')|f(r\xi')|^p dV_{\xi'}.$$

For $\xi\in D$, then exists a holomorphic automorphism (say γ_ξ) such that $\gamma_\xi(z_0)=0$ which maps ξ into ξ'; then $T(z, \bar{\xi})dV_\xi = \frac{1}{V}dV_{\xi'}$. Consequently

$$\frac{1}{V}\int_D |f(r\xi')|^p dV_{\xi'} = \int_D T(z_0, \bar{\xi})|f(r\xi)|^p dV_\xi \le C(z_0).$$

Hence

$$\sup_{0\le r<1}\left(\frac{1}{V}\int_D |f(r\xi')|^p dV_{\xi'}\right) \le C(z_0) < \infty,$$

and $f\in A^p(D)$.

The following generalizes a resutl for H^p spaces [2, p. 92]. We have:

THEOREM 4. *Let D be a bounded symmetric domain and $\{f_n\}$ be a bounded* *sequence in* $A^p (p>0)$. *Then* $f_n \to f$ *pointwise on* D *if and only if* $f_n \to f$ *uniformly on compact subsets of* D; *also* $f \in A^p$.

PROOF. Since $\{f_n\}$ is bounded in $A^p (p>0)$, by (3.3) and Vitali's Convergence Theorem for C^n [1, Lemma 4], $f_n \to f$ pointwise on D implies that $\{f_n\}$ converges uniformly to f on compact subsets of D.

Conversely, if $f_n \to f$ uniformly on compact subsets of D, then $f_n \to f$ pointwise on D. We have left to prove that $f \in A^p$.

By uniform convergence on $\bar{D}_r$ and hence on D_r for $0 \le r < 1$

$$\lim_{n\to\infty} \int_{D_r} |f_n(z)|^p dV_z = \int_{D_r} |f(z)|^p dV_z.$$

Now, since $\{f_n\}$ is bounded in $A^p (p>0)$, there exists a constant C, independent of n, such that

$$\|f_n\|_{A^p} \le C,$$

and thus

$$\int_{D_r} |f(z)|^p dV_z = \lim_{n\to\infty} \int_{D_r} |f_n(z)|^p dV_z$$
$$\le \lim_{n\to\infty} \int_D |f_n(z)|^p dV_z \le C,$$

which implies that

$$r^{2N} \int_D |f(rz)|^p dV_z \le C.$$

Fix r in $[0, \frac{1}{2}]$. Then f_r is continuous on $\bar{D}$ and consequently bounded on $\bar{D}$. Therefore

$$\sup_{0\le r\le \frac{1}{2}} \int_D |f(rz)|^p dV_z < \infty. \tag{3.4}$$

For r in $[\frac{1}{2}, 1)$

$$\int_D |f(rz)|^p dV_z \le 2^N C,$$

and consequently

$$\sup_{\frac{1}{2}\le r<1} \int_D |f(rz)|^p dV_z < \infty.$$

Hence from this result and (3.4)

$$\sup_{0\le r<1} \int_D |f(rz)|^p dV_z < \infty, \text{ and } f \in A^p.$$

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