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ON THE INTEGRAL PROPERTIES OF CLASSICAL SOLUTIONS OF A PARABOLIC EQUATION

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1. Introduction

Questions of classical and weak solutions and their regularity of linear second-order parabolic equations with different boundary conditions are extentively studied in [1], [2], [3], etc. It is well known that a situation can arise, in which the first mixed boundary problem for such an equation has a classical solution, but does not have a weak one. The classical solution is, at the same time, a weak one if it is subject to additional conditions of integral or differential character. We aim at investigating the conditions that must be imposed on the known functions of the problem, that is, the coefficients, right-hand side and the functions defining the initial and boundary conditions, in order that a classical solution of the problem has square integrable derivatives with respect to space and time variables. No conditions will be imposed on the lateral surface of the region. We note that similar ideas for the case of elliptic equations were considered in [4], [5] and [6].

In the Euclidean space R^{n+1} of points $(x, t)=(x_1, \cdots, x_n, t)$ we consider the cylinderical region $Q=G\times(0, T)$ of height $T>0$, where G is a bounded or unbounded region in R^n with boundary ∂G. Let $S=\partial G\times(0, T)$ and $G_c=\{(x, t)\in Q : t=c\}$.

Throughout this paper, we shall employ usual notation for spaces of continuous and continuously differentiable functions and for Sobolev spaces [2]. $V^{1,0}(Q)$ denotes the Banach space of all functions in $W^{1,0}(Q)$, which are continuous with respect to t in the $L^2(G)$-norm, that is, $\lim_{h\to 0} \|u(x, t+h)-u(x, t)\|_{L^2(G)}=0$, and such that

$$\|u\|_{V^{1,0}(Q)}=\max_{0\le t\le T}\|u(x, t)\|_{L^2(G)}+\|Du\|_{L^2(Q)}<\infty,$$

where $D=(D_1, \cdots, D_n)$ and $D_i=\partial/\partial x_i$, $i=1, \cdots, n$, and $|Du|^2=\sum_{i=1}^{n}|D_iu|^2$.

In Q consider the differential operator L defined by

$$Lu = D_t u - D_i(a_{ij} D_j u) + a_i D_i u + au,$$

where $D_t = \partial/\partial t$ and summation convention is understood. All coefficients of L are assumed to be continuous functions in Q, and L is strictly parabolic in Q, that is,

$$\nu|\xi|^2 \le a_{ij}\xi_i\xi_j \le \mu|\xi|^2, \quad \nu,\ \mu = \text{const} > 0, \tag{1}$$

for any real $\xi \in R^n$ and all $(x,\ t) \in Q$.

From here and on, $u(x,\ t)$ will denote a classical solution from the space $C^{2,1}(Q) \cap C(\bar{Q})$ of the problem

$$\left.\begin{aligned} &Lu = f(x,\ t) \text{ in } Q, \\ &u|_{t=0} = g(x),\ u|_s = 0, \end{aligned}\right\} \tag{2}$$

where $f \in C(Q)$ and $g \in C(\bar{G})$. Clearly, $u \in L^2(Q)$ in case of bounded Q. If Q is unbounded, we have $u \in L^2[G' \times (0,\ T)]$ for any $G' \subset\subset G$.

2. Integrability of *Du* on interior cylinders of height *T*

In this section we prove the following auxiliary result.

LEMMA 1. *Let the operator L be strictly parabolic in Q and have coefficients a_i, $a \in L^\infty(Q)$, and let $f \in L^2(Q)$. If problem* (2) *has a classical solution u, then for any subregion $G' \subset\subset G$, we have $u \in V^{1,0}(Q')$, where $Q' = G' \times (0,\ T)$.*

PROOF. Let $q(x) \in C^1(\bar{G})$ be a cut-off function with compact support in G and such that $0 \leqq q \leqq 1$ and $q = 1$ on G'. Multiply both sides of the equation $Lu = f$ by $q^2 u$ and integrate the result over the cylinder $Q_{t_1, t_2} = G \times (t_1,\ t_2)$, where $0 < t_1 < t_2 < T$. Properties of q and u enable us to apply the formula of integration by parts to obtain

$$\frac{1}{2}\int_{G_{t_2}} q^2 u^2 dx - \frac{1}{2}\int_{G_{t_1}} q^2 u^2 dx + \int_{Q_{t_1, t_2}} [a_{ij} D_j u D_i(q^2 u) + a_i q^2 u D_i u + a q^2 u^2] dx dt = \int_{Q_{t_1, t_2}} q^2 f u dx dt.$$

Using (1), Cauchy inequalities $|ab| \le \varepsilon a^2 + (1/4\varepsilon) b^2$ and $|a_{ij}\alpha_i\beta_j| \le (a_{ij}\alpha_i\alpha_j)^{\frac{1}{2}}(a_{ij}\beta_i\beta_j)^{\frac{1}{2}}$, and Schwarz inequality, we obtain

$$\frac{1}{2}\int_{G_{t_2}} q^2 u^2 dx + (\nu - \varepsilon_1 - \varepsilon_2)\int_{Q_{t_1, t_2}} q^2 |Du|^2 dx dt \le \frac{1}{2}\int_{G_{t_1}} q^2 u^2 dx + \frac{1}{2}\int_{Q_{t_1, t_2}} q^2 f^2 dx dt + \frac{2}{2\varepsilon_1}\int_{Q_{t_1, t_2}} |Dq|^2 u^2 dx dt + \Big[\frac{1}{2} + \sup_Q \Big(a + \frac{1}{4\varepsilon_2}\sum_{i=1}^n a_i^2\Big)\int_{Q_{t_1, t_2}} q^2 u^2 dx dt.$$

Choosing ε_1 and ε_2 sufficiently small such that $\nu-\varepsilon_1-\varepsilon_2>0$, and passing to the limit as $t_1\to 0$ in this inequality, we get

$$\int_{Q_{0,t_2}} q^2|Du|^2dxdt\le C\Big(\int_Q [q^2f^2+q^2u^2+|Dq|^2u^2]dxdt+\int_G q^2g^2dx\Big)<\infty,$$

where C depends only on ν, μ and the coefficients of L, and is independent of t_2. Hence, $q|Du|\in L^2(Q)$, which together with the fact that $u\in C(\bar{Q}')$, completes the proof of the lemma.

3. Main result

Let G_m, $m=1,2,\cdots$, be a sequence of subregions of G, for which $G_m\subset\subset G_{m+1}$, ∂G_m are sufficiently smooth and $\lim\limits_{m\to\infty} G_m=G$. Since $u\in C(\bar{Q})$ and $u=0$ on S, we can, in the case of bounded G, choose these G_m in such a way that

$$|u|<\frac{1}{m} \text{ for all } (x,\ t)\in\{G\backslash G_m\}\times[0,\ T]. \tag{3}$$

The fulfilment of (3) in the case of unbounded G may be achieved by additionally assuming that

$$\lim_{|x|\to\infty} u(x,\ t)=0 \text{ and } u\in L^2(Q). \tag{4}$$

In the cylinder $Q_m=G_m\times(0,\ T)$, whose lateral surface we denote by S_m, consider the problem

$$\left.\begin{aligned} &Lu_m=f \text{ in } Q_m,\\ &u_m|_{t=0}=h_m(x)g,\ u_m|_{S_m}=0,\end{aligned}\right\} \tag{5}$$

where $h_m\in C^1(\bar{G})$, $0\le h_m\le 1$, $h_m=1$ on G_{m-1} and supp $h_m\subset\subset G_m$. If such u_m exists, we denote by u_m^* its extension to the whole Q, which is equal to zero in $Q\backslash Q_m$.

LEMMA 2. *Let us assume in addition to the hypotheses of lemma* 1, *that* $g\in\overset{\circ}{W}{}^1(G)$ *and* $\int_0^T$ *ess* $\sup\limits_G|D_ta_{ij}|dt<\infty$. *Then* $u_m^*\in\overset{\circ}{W}{}^{1,1}(Q)$ *and there is a subsequence of* $\{u_m^*\}^G$*weakly converging in the metric of* $W^{1,0}(Q)$ *to a weak solution* u^* *in* $\overset{\circ}{W}{}^{1,0}(Q)$ *of problem* (2). *Moreover,*

$$\|u^*\|_{W^{1,0}(Q)}\le C(\|f\|_{L^2(Q)}+\|g\|_{L^2(G)}), \tag{6}$$

where C *is independent of* f *and* g.

PROOF. Properties of h_m and g imply $h_mg\in\overset{\circ}{W}{}^1(G_m)$. In view of this and our conditions on the coefficients and f, it follows from the well known existence theorem for parabolic equations; cf [2, Ch. III], that problem (5) possess a weak solution $u_m \in \overset{\circ}{W}{}^{1,1}(Q_m)$. This solution is understood to be a function $u_m \in \overset{\circ}{W}{}^{1,1}(Q_m)$ satisfying the initial condition in (5) and the integral identity

$$\int_{Q_m} (vD_t u_m + a_{ij} D_j u_m D_i v + a_i v D_i u_m + a u_m v)\, dxdt = \int_{Q_m} fv dxdt$$

for any $v \in \overset{\circ}{W}{}^{1,1}(Q_m)$. In this identity set $v = u_m e^{-kt}$, where k is a positive constant to be determined later. Using (1), integration by parts and Cauchy inequality, we get

$$\frac{\nu}{2} e^{-kT} \int_{Q_m} |Du_m|^2 dxdt + \left(\frac{k}{2} + \inf_Q a - \frac{1}{2\nu} \sup_Q \sum_{i=1}^n a_i^2 - \frac{1}{2}\right) \int_{Q_m} e^{-kt} u_m^2 dxdt$$
$$\leq \frac{1}{2} \int_{Q_m} f^2 dxdt + \frac{1}{2} \int_{G_m} h_m^2 g^2 dx.$$

Choosing k sufficiently large such that

$$\frac{k}{2} + \inf_Q a - \frac{1}{2\nu} \sup_Q \sum_{i=1}^n a_i^2 - \frac{1}{2} > 0$$

and recalling the properties of h_m, we have from last inequality

$$\int_{Q_m} (|Du_m|^2 + u_m^2)\, dxdt \leq C(\|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(G)}),$$

where C is independent of m, which shows that

$$\|u_m^*\|_{W^{1,0}(Q)} \leq C(\|f\|_{L^2(Q)} + \|g\|_{L^2(G)}) \leq K, \tag{7}$$

with K independent of m. Hence, there is a subsequence of $\{u_m^*\}$ weakly converging in the metric of $\overset{\circ}{W}{}^{1,0}(Q)$ to some function u^* in $\overset{\circ}{W}{}^{1,0}(Q)$. Properties of h_m and (3) imply

$$|h_m g - g| = (1 - h_m)|g| < \frac{1}{m-1} \text{ for all } x \in G.$$

Thus, the traces $h_m g$ of u_m^* on G_0 uniformly converge to g in G_0. Hence, $u^*|_{t=0} = g$. By elementary considerations, we may show that u^* is a weak solution of problem (2) in the class $\overset{\circ}{W}{}^{1,0}(Q)$.

Now, we formulate and prove our main result.

THEOREM. *Let all the hypotheses of lemma 2 be satisfied. If problem (2) has a classical solution u, which in case of unbounded G satisfies the extra conditions (4), then $u \in \overset{\circ}{W}{}^{1,1}(Q)$ and, hence, coincides with the weak solution of problem (2) in $\overset{\circ}{W}{}^{1,1}(Q)$.*

PROOF. In Q_m consider the problem

$$\left.\begin{aligned} &Lw_m+pw_m=f+pu \quad \text{in } Q_m, \\ &w_m|_{t=0}=h_m g, \ \ w_m|_{S_m}=0, \end{aligned}\right\}$$

where p is some fixed number, for which $p+\inf\limits_Q a>0$. By lemma 2, we have $w_m \in \mathring{W}^{1,1}(Q_m)$ and there is a subsequence of $\{w_m^*\}$ weakly converging in the metric of $W^{1,0}(Q)$ to the weak solution u^* of problem (2). Since, under our assumptions, a solution in $W^{1,0}(Q)$ is also a solution in $W^{1,1}(Q)$, we have $u^* \in \mathring{W}^{1,1}(Q)$. Without loss of generality, we can assume that the sequence $\{w_m^*\}$ itself weakly converges in $W^{1,0}(Q)$-metric to u^*.

Lemma 1 and properties of w_m imply that $u-w_m$ is a weak solution in $V^{1,0}(Q_m)$ of the problem

$$\left.\begin{aligned} &Lv_m+pv_m=0 \ \ \text{in } Q_m, \\ &v_m|_{t=0}=(1-h_m)g, v_m|_{S_m}=u_{S_m}. \end{aligned}\right\}$$

Taking into consideration our conditions concerning the coefficients of L and the fact that $p+a>0$ in Q, we may apply the maximum principle for solutions of parabolic equations, as stated in theorem 7.2 in [2, Ch. III], to obtain in view of (3) that

$$|u-w_m| \leq \sup_{S_m \cup \overline{G}_{m_0}} |u-w_m| \leq \frac{1}{m-1}$$

almost everywhere (a.e.) in Q_m, where $G_{m_0}=\{(x, t): x \in G_m, \ t=0\}$. This inequality, together with (3), shows that

$$|u-w_m^*|<\frac{1}{m-1} \text{ a.e. in } Q.$$

Hence, w_m^* uniformly converges to u a.e. in Q. But we showed above that the same sequence weakly converges in $W^{1,0}(Q)$-metric to u^*, so we must have $u=u^*$ a.e. in Q. Therefore, u is the weak solution in $\mathring{W}^{1,1}(Q)$ of problem (2) and the proof is complete.

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