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ISOMETRIC IMMERSION INTO A HYPERBOLIC SPACE

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1. Introduction

In this work we discuss the problem of isometric immersion $x: M \to H^{n+N}$ of a compact, connected, oriented n-dimensional C^{∞} Riemannian manifold M into the $(n+N)$-dimensional hyperbolic space H^{n+N}. The same problem has been studied in the $(n+N)$-Euclidean space E^{n+N} [3] and also in the unit $(n+N)$-dimensional sphere S^{n+N} [1] as ambient spaces ($n \geq 2$, $N \geq 1$).

Let $B(M)$ be the bundle of unit normal vectors to $x(M)$, i.e., a point of $B(M)$ is a pair $(p, \nu(p))$, where $\nu(p)$ is a unit normal vector to $x(M)$ at $x(p)$.

Actually, in [3] and [1] the following two theorems have been proved.

THEOREM 1 [3]. *Assume all second quadratic forms of the immersion* $x: M \to E^{n+N}$ *to be semi-definite and definite at one point* $(p, \nu(p)) \in B(M)$. *Then* $x(M)$ *belongs to a linear subvariety* E^{n+1} *of* E^{n+N} *and* x *imbeds* M *as the boundary of a convex body; in particular* M *is homeomorphic to a sphere.*

THEOREM 2 [1]. *Let* $x: M \to H_s^{n+N}$ *be a locally convex isometric immersion of a compact, connected, oriented, n-dimensional* C^{∞} *Riemannian manifold* M *into the open hemisphere* $H_s^{n+N} \subset S^{n+N}$. *Then* $x(M)$ *belongs to a totally geodesic sphere* $S^{n+1} \subset S^{n+N}$ *and* x *imbeds* M *as the boundary of a convex body of* S^{n+1}. M *is homeomorphic to a sphere.*

The aim of this paper is to prove the following theorem:

THEOREM 3. *Consider* $x: M \to H^{n+N}$ *to be a locally convex isometric immersion of a compact, connected, oriented n-dimensional* C^{∞} *Riemannian manifold* M *then* $x(M)$ *belongs to an* $(n+1)$*-totally geodesic submanifold* H^{n+1} *of* H^{n+N} *and* x *imbeds* M *as the boundary of a convex body of* H^{n+1}. M *is homeomorphic to a sphere.*

Aming to our study we give some definitions and basic materials in the following section.

2. Basic materials

i) Convex bodies

A subset B in a Riemannian manifold M is called convex if for each pair of points p, $q \in B$, there is a unique minimal geodesic segment from p to q and this segment is in B. An open (closed) convex set which is a submanifold of M of maximal dimension is called open (closed) convex body.

ii) Convex submanifolds

Let M be a submanifold of a Riemannian manifold $\bar{M}$. We say that M is convex at the point $p \in M$ in the direction of ν where $\nu \in M_p^{\perp} - M_p$ means the tangent space of M at $p \in M$ (or simply M is convex at the point $(p, \nu) \in B(M)$) if there is a neighbourhood of p in M which lies on one side of the geodesic hypersurface of $\bar{M}$ at p generated by the hyperplane W of $\bar{M}_p$ with ν as its unit normal. We denote such a hypersurface by $H(p, \nu) = \exp_p W$.

The submanifold M will be called convex at the point $p \in M$ if it is convex at each point $(p, \nu) \in B(M)$ in the fiber over p. M is locally convex if it is convex at each point $(p, \nu) \in B(M)$ in the normal bundle.

In a similar way we can define the convexity of M at $(p, \nu) \in B(M)$ by requiring that M lies-as a whole-on one side of the geodesic hypersurface of $\bar{M}$ at p generated as mentioned above.

iii) Height function and second fundamental form

By the height function L_p^{ν} for an oriented submanifold M at the point $(p, \nu) \in B(M)$ we mean the function defined on a sufficiently small neighbourhood of the origin in the tangent space of the submanifold at p and assigning to each point of this neighbourhood the height, with respect to ν, of the submanifold above the tangent hyperplane $W = H_p(p, \nu)$ as measured in the ambient manifold. Consequently,

$$L_p^{\nu}(x) = \langle y, \nu \rangle$$

where $y = \exp_p^{-1} x$ [4].

Let M be a submanifold of dimension n in a Riemannian manifold $\bar{M}$ of dimension $(n+N)$. Let $\bar{\nabla}$ denote the Riemannian connection of $\bar{M}$, ∇ the induced connection on M and $\langle\, ,\, \rangle$ the Riemannian metric of $\bar{M}$. The second fundamental form $\alpha : M_p \times M_p \to M_p^{\perp}$ of M at the point $p \in M$ is a bilinear map defined by $\alpha(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$ for vector fields X, Y tangent to M. The second fundamental form $\alpha^{\nu}: M_p \times M_p \to \boldsymbol{R}$ in the direction $\nu \in M_p^{\perp}$ may be taken as

$$\alpha^{\nu}(X,\ Y)=\langle \alpha(X,\ Y),\ \nu\rangle.$$

By second quadratic form q_{ν} of M at $p\in M$ in the ν-direction we mean the map $q_{\nu}: M_p \to \boldsymbol{R}$ defined by restriction as

$$q_{\nu}(X)=\alpha^{\nu}(X,\ X)=\langle \alpha(X,\ X),\ \nu\rangle.$$

Let p be a point of M. It is possible to take a system of normal coordinates $y^1,\ y^2, \cdots, y^{n+N}$ with origin p such that $\left(\frac{\partial}{\partial y^1}\right)_p, \cdots, \left(\frac{\partial}{\partial y^n}\right)_p$ span M_p. Let $\{Y_i\}_{i=1}^{n+N}$ be an orthonormal basis of M_p such that $Y_1, \cdots, Y_n$ form a basis of M_p. We may choose the system of normal coordinates $y^1, \cdots, y^{n+N}$ such that $\left(\frac{\partial}{\partial y^i}\right)_p = Y_i$, $1\leq i \leq n+N$. Note that $Y_{n+1}, \cdots, Y_{n+N}$ form a basis of the normal subspace $M_p^{\perp}\subset \bar{M}$.

Let $x^1, \cdots, x^n$ be an arbitrary coordinate system in a neighbourhood U of p in M and let

$$y^i = y^i\ (x^1, \cdots, x^n)\ \ 1\leq i \leq n+N$$

be the system of equations that defines the imbedding of U into $\bar{M}$. The second fundamental form of the above embedding is shown to be [5]

$$\alpha\Big(\Big(\frac{\partial}{\partial x^{\lambda}}\Big)_p,\ \Big(\frac{\partial}{\partial x^{\mu}}\Big)_p\Big)=\sum_{k=n+1}^{n+N}(\partial^2 y^k/\partial x^{\lambda} x^{\mu})_p\ Y_k$$

Namely, the coefficients of α_p with respect to the basis $\left(\frac{\partial}{\partial x^1}\right)_p, \cdots, \left(\frac{\partial}{\partial x^n}\right)_p$ in M_p and the basis $Y_{n+1}, \cdots, Y_{n+N}$ in $M_p^{\perp}$ are those of the hessian $(\partial^2 y^k/\partial x^{\lambda} x^{\mu})$ at p.

From the last equation, we see that the second fundamental form of the embedding considered above in the direction of $\nu \in M_p^{\perp}$ (ν is a unit normal) is given by

$$\begin{aligned}\alpha^{\nu}\Big(\Big(\frac{\partial}{\partial x^{\lambda}}\Big)_p,\ \Big(\frac{\partial}{\partial x^{\mu}}\Big)_p\Big)&=\Big\langle \alpha\Big(\Big(\frac{\partial}{\partial x^{\lambda}}\Big)_p,\ \Big(\frac{\partial}{\partial x^{\mu}}\Big)_p\Big),\ \nu\Big\rangle\\ &=\sum_{k=n+1}^{n+N}(\partial^2 y^k/\partial x^{\lambda}\partial x^{\mu})_p\ \langle Y_k,\ \nu\rangle\\ &=\sum_{k=n+1}^{n+N}\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}\ y^k(p)\ \langle Y_k,\ \nu\rangle\\ &=\sum_{k=1}^{n+N}\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}\langle y^k(p)\ Y_k,\ \nu\rangle=\frac{\partial^2 L^{\nu}{}_p(p)}{\partial x^{\lambda}\partial x^{\mu}}.\end{aligned}$$

Now, we arrived at the following lemma 1 :

LEMMA 1. *Let $x : M \to \bar{M}$ be an immersion of a manifold M into a Riemannian manifold $\bar{M}$. For a point $p \in M$, let U be the normal coordinate neighbourhood of M around the point $x(p)$. Let L_p^ν be the height function of M in the direction of $\nu \in M_p^\perp$ at p as being defined before. Then L_p^ν has a critical point at p and its Hessian form is the second fundamental form at p in the ν-direction.*

It is an easy geometric exercise to conclude that:

COROLLARY 1. *Let $x : M \to \bar{M}$ be a locally convex immersion of a manifold M into a Riemannian manifold $\bar{M}$, then all second quadratic forms of the immersion x are semi-definite.*

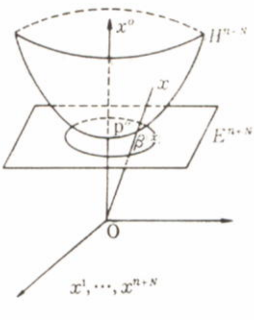
Now, if M is a compact n-submanifold of a Euclidean $(n+N)$-space, then there is an $(n+N-1)$-sphere $S^{n+N-1}(k)$ with finite radius k which contains M inside. Let $p \in M \cap S^{n+N-1}(k)$ and ν be a unit normal to the sphere at p. Clearly, ν is also normal to M at p and p is a non-degenerate critical point of the height function L_p^ν on M. In fact, p is either maxima or minima. This discussion together with lemma 1 show that the second quadratic form q_ν of M at p in the ν-direction is definite. Consequently, we have an important result stating that the definitness condition in theorem 1 is unnecessary.

iv) Beltrami map

The central projection or Beltrami map we are going to discuss, represents an important tool for proving our theorem 3 by transfering the immersion problem under consideration from hyperbolic space H^{n+N} to E^{n+N}. We take the sphererical model as a model for hyperbolic space H^{n+N} which is defined as

$$H^{n+N} = \{(x^0, \cdots, x^{n+N}) \in V^{n+N+1} : -(x^0)^2 + (x^1)^2 + \cdots + (x^{n+N})^2 = -1\}$$

where V^{n+N+1} denote the Minkowski space $(R^{n+N+1}, \langle\,,\rangle)$ with the metric $\langle\,,\rangle = -dx^0 \otimes dx^0 + \sum_{i=1}^{n+N} dx^i \otimes dx^i$.



The Beltrami map (central projection) [4] $\beta : H^{n+N} \to E^{n+N}$ is defined to be the map which takes $x \in H^{n+N}$ to the intersection of E^{n+N}-defined by $x^0 = 1$—with the straight line through x and the origin 0 of V^{n+N+1} as indicated in the figure.

In the usual coordinates $x = (x^0, x^1, \cdots, x^{n+N})$, the map β may be expressed mathematically as follows

$$\beta(x)=x/x^0=\left(1,\ \frac{x^1}{x^0},\ \cdots,\frac{x^{n+N}}{x^0}\right).$$

The map β takes H^{n+N} diffeomorphically to the open disk $D(p,\ 1)\subset E^{n+N}$ of center p and radius 1. From the geometric point of view β is a geodesic map. Consequently, β takes a totally geodesic m-submanifold of H^{n+N} to a linear subvariety E^m of E^{n+N}. Also, if a submanifold $M\subset H^{n+N}$ is locally convex then so is $\beta(M)\subset E^{n+N}$.

COROLLARY 2. *If $M\subset H^{n+N}$ is locally convex then all second quadratic forms of $\beta(M)$ are semi-definite.*

3. Proof of theorem 3

Now, we are going to prove theorem 3 after the above discussion. It is clear that all the conditions of theorem 1 are satisfied for the immersion $\beta\circ x$: $M\to E^{n+N}$. Consequently, $\beta\circ x(M)$ belongs to a linear subvariety $E^{n+1}\subset E^{n+N}$ and $\beta\circ x$ is an embedding. Since β is a diffeomorphism, then the map x should be an embedding as well. In addition $\beta\circ x(M)$ forms a boundary of a convex body in E^{n+1} and M is homeomorphic to a sphere.

The linear subvariety E^{n+1}-in which M is embedded by $\beta\circ x$-is taken to a totally geodesic submanifod $H^{n+1}\subset H^{n+N}$ of dimension $(n+1)$ under the map β^{-1}.

Now, we show that $x(M)$ bounds a convex body B in H^{n+1}. Assume, on the contrary, that B is not convex, then there are either

(i) More than a minimal geodesic segment between a pair of points $p,\ q\in B$.
or (ii) A unique minimal geodesic segment from p to q which is not in B.

Case (i) is not true since H^{n+1} has no conjugate points i.e. all its points are poles, then there is a unique geodesic through any two points of H^{n+1}.

Case (ii) if $p,\ q\in B$ have a unique minimal geodesic segment γ, say, which is not in B and included in H^{n+1}. Under β, which is a geodesic map, the body $\beta(B)$ would not be a convex body in E^{n+1} which is a contradiction. Thus B is a convex body in H^{n+1} and the theorem is proved.

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