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THE T_1-CONTINUOUS FUNDAMENTAL GROUP OF A CERTAIN FINITE SPACE

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1. Introduction

Let X be a topological space and let $x_0 \in X$. Then $C(X,\ x_0)$ will be used to denote the set of all continuous loops in X at x_0. The idea of using continuous functions as relating functions on $C(X,\ x_0)$ to get an equivalence relation on $C(X,\ x_0)$ has long been in existence, and extensive studies have been made of the resulting homotopy groups. In [5], we considered using certain types of non-continuous functions as relating functions on $C(X,\ x_0)$. In particular an admitting homotopy relation N was defined, which in general, turned out to be a larger class of relating functions than the class of continuous functions. Most types of non-continuous functions, including almost continuous functions [1], C-continuous functions [2], connectivity maps [6], and T_1-continuous functions [4], provide an admitting homotopy relation. Also in [5], it was shown how an admitting homotopy relation N could be used to obtain a generalized homotopy group $N(X,\ x_0)$. The question has been raised as to an example of when one of these generalized homotopy groups is different from the corresponding usual homotopy group. In this paper we let N be the admitting homotopy relation T_1-continuous and give an example of a space X and a point $x_0 \in X$ such that the T_1-continuous fundamental group $N(X,\ x_0)$ is different from the fundamental group $\Pi_1(X,\ x_0)$. That is if the relating functions between the loops are only required to be T_1-continuous, then we get a different group than if we required the relating functions between the loops to be continuous.

Throughout this paper I will be used to denote the closed unit interval with the usual topology.

2. The example

EXAMPLE. Let $X=\{a,\ b,\ c,\ d\}$, and let $T=\{\phi,\ X,\ \{b\},\ \{c\},\ \{b,\ c\},\ \{a,\ b,\ c\},\ \{b,\ c,\ d\}\}$. Then $\Pi_1(X,\ b)$ is not isomorphic to $N(X,\ b)$.

PROOF. Let $f: I \to X$ be the continuous function defined by $f(x)=b$ for all $x \in I$ and let $g: I \to X$ be a continuous function such that $g(0)=b=g(1)$. Then since g is continuous and $\{a, b, c\} \in T$, $g^{-1}(\{a, b, c\})$ is open in I and thus $D=\{x \mid g(x)=d\}$ is closed in I. Similarly, $A=\{x \mid g(x)=a\}$ is closed in I.
Define $F: I \times I \to X$ by

$$F(x, t)=\begin{cases} d \text{ if } x \in D \text{ and } 0 \leq t \leq 1/2 \\ a \text{ if } x \in A \text{ and } 0 \leq t \leq 1/2 \\ g(x) \text{ if } t=0 \\ b \text{ otherwise} \end{cases}$$

Then F is well-defined and clearly $F(0, t)=b=F(1, t)$ for all $t \in I$ and $F(x, 0)=g(x)$ and $F(x, 1)=f(x)$ for all $x \in I$. We wish to show that F is T_1-continuous. Let $\mathscr{U}$ be an open cover of X. Then either $X \in \mathscr{U}$ or $\{a, b, c\}$ and $\{b, c, d\}$ are in $\mathscr{U}$. If $X \in \mathscr{U}$, then an open cover of $I \times I$ which will work is $\{I \times I\}$. If $\{a, b, c\}$ and $\{b, c, d\}$ are in $\mathscr{U}$, then an open cover of $I \times I$ which will work is $\{I \times I - D \times [0, 1/2], I \times I - A \times [0, 1/2]\}$. Hence, F is T_1-continuous. It follows that $N(X, b)$ is the trivial group.

We will now show that $\Pi_1(X, b)$ has at least two elements. Once again let f be the constant loop at b and define $h: I \to X$ by

$$h(x)=\begin{cases} b \text{ if } 0 \leq x < 1/5 \\ a \text{ if } 1/5 \leq x \leq 2/5 \\ c \text{ if } 2/5 < x < 3/5 \\ d \text{ if } 3/5 \leq x \leq 4/5 \\ b \text{ if } 4/5 < x \leq 1 \end{cases}$$

Now f and h are loops at b and we wish to show that f and h are not homotopic modulo b. To this end suppose that there is a continuous function $F: I \times I \to X$ such that $F(x, 0)=h(x)$, $F(x, 1)=f(x)$, and $F(0, t)=b=F(1, t)$ for all $x \in I$, $t \in I$. Let p and q be the points $p=(2/5, 0)$, $q=(3/5, 0)$. Let $J=(2/5, 3/5) \times \{0\}$. Since $\{c\} \in T$, $f^{-1}(\{c\})$ is an open subset of $I \times I$. Since $F(x, 0)=h(x)$ for all $x \in I$, $F^{-1}(\{c\})$ contains J. Let U be the component of $F^{-1}(\{c\})$ which contains J. Then U is open and connected and since F is h on $I \times \{0\}$, f on $I \times \{1\}$, and b on $\{0\} \times I$ and $\{1\} \times I$, the only points on the boundary of $I \times I$ which are in U are in J. Let B be the boundary of U. Let $W=I \times I - \bar{U}$ and let $M=W \cup B \cup J$. Then $W \cup B$ is closed in $I \times I$ and since $p, q \in B$, $W \cup B \cup J$ is closed. Hence, M is closed. Since J is the intersection of the boundary of $I \times I$ and U, the boundary of $I \times I$ is contained in M. Let Q be the component of M which contains the boundary of $I \times I$. Then Q is closed and connected. Since Q is bounded, Q is compact and hence a continuum. Since $I\times I$ is closed in the plane, Q is a continuum in the plane. Since J is a subset of the boundary of $I\times I$ and U is an open, connected subset of $I\times I$ containing J, $U-J$ is connected. Now $U-J$ is a connected subset of the compliment of Q. Let $\mathcal{O}$ be the component of the compliment of Q which contains $U-J$.

We wish to show that the boundary of $\mathcal{O}$ is a subset of J union the boundary of U. Let $x\in \text{bd }\mathcal{O}$. Then $x\in M$ and thus $x\in W\cup B\cup J$. If $x\in B\cup J$, then clearly $x\in(\text{bd }U)\cup J$. Now suppose $x\in W$. Since W is an open subset of $I\times I$, there is a disc D in the plane such that $x\in D\cap(I\times I)\subset W$. Now $x\in\text{bd }\mathcal{O}$ and thus $x\in Q$. But since D is connected and contains x and Q is the component containing x, $D\cap(I\times I)\subset Q$. Now Q contains the boundary of $I\times I$ and $\mathcal{O}$ is a component of the compliment of Q which intersects the interior of $I\times I$. Hence, $\mathcal{O}$ is contained in the interior of $I\times I$ and thus x is neither a point nor a limit point of $\mathcal{O}$. Therefore, $x\notin\text{bd }\mathcal{O}$. But this is impossible. Hence, $x\notin W$. Thus, $\text{bd }\mathcal{O}\subset J\cup(\text{bd }U)$. By [12, Theorem 2.1, p. 105], since $\mathcal{O}$ is a bounded component of the compliment of Q, the bd $\mathcal{O}$ is a continuum. Let K be the boundary of $\mathcal{O}$. Let $L=K-J$. Then $L\subset\text{bd }U$ and we now wish to show that L is connected. Since p, $q\in K$ and neither p nor q is in J, p, $q\in L$. Suppose L is not connected. Then L is the union of two non-empty, mutually separated sets $\mathscr{A}$ and $\mathscr{B}$ with p in one of them. Say $p\in\mathscr{A}$. Suppose $q\in\mathscr{A}$. Then $K=(\mathscr{A}\cup J)\cup\mathscr{B}$. Now $\mathscr{A}$ and $\mathscr{B}$ are mutually separated. Since $\mathcal{O}$ is an open subset of $I\times I$ containing J in its boundary, no point of J is a limit point of $K-J$ and no point of $K-J$ is a limit point of J except p and q. But p and q are in $\mathscr{A}$. Hence, J and $\mathscr{B}$ are mutually separated. Thus, $\mathscr{A}\cup J$ and $\mathscr{B}$ are non empty, mutually separated sets. But this is impossible, since K is connected. Thus, $q\in\mathscr{B}$. Now suppose $\mathscr{A}$ is not connected. Then $\mathscr{A}=\alpha\cup\beta$ where α and β are non-empty mutually separated sets with $p\in\alpha$. Then $K=\beta\cup(\alpha\cup J\cup\mathscr{B})$ where these two sets once again are mutually separated. Thus, $\mathscr{A}$ is connected. Since J is an open subset of K, $K-J$ is closed and thus L is closed. Since $\mathscr{A}$ is a component of L, $\mathscr{A}$ is closed. Hence, $\mathscr{A}$ is a continuum. Similarly, $\mathscr{B}$ is a continuum. By [12, Theorem 3.1, 108], there is a simple closed curve Γ in the plane such that Γ separates p from q and $\Gamma\cap(\mathscr{A}\cup\mathscr{B})=\phi$. Let Z be the boundary of $I\times I$ minus $J\cup\{p, q\}$. Then $J\cup\{p, q\}$ is a connected set containing p and q and since Γ separates p from q, $\Gamma\cap J\neq\phi$. Let $w\in\Gamma\cap J$. Similarly $\Gamma\cap Z\neq\phi$. Let $z\in\Gamma\cap Z$. Since $z\in\Gamma\cap Z$, there is a point k in the unbounded component of the compliment of the boundary of $I\times I$ such that $k\in\Gamma$ and the arc from k to z in Γ not containing

w contains no point of J. Since J is in the boundary of $\mathcal{O}$ there is a point $m \in \mathcal{O}$ such that $m \in \Gamma$ and the arc from k to m in Γ containing z contains no point of J. Let Λ be the arc in Γ from k to m containing z. Then $\Lambda \cap J = \phi$ and since $\Gamma \cap (\mathscr{A} \cup \mathscr{B}) = \phi$, $\Lambda \cap K = \phi$. But then the component of the compliment of K containing $\mathcal{O}$ is not a subset of the interior of $I \times I$, which is impossible. Hence, L is connected. Since $L = K - J$ and $K \subset (\text{bd } U) \cup J$, $L \subset \text{bd } U$. Hence a connected subset of the boundary of U contains both p and q.

Let P be the component of the boundary B of U which contains p and q. Then since B is closed, P is closed.

Now U was the component of $F^{-1}(\{c\})$ containing J. Thus, no point of B is in $F^{-1}(\{c\})$, for if $x \in B$ and $F(x) = c$, then since F is continuous at x, there is a disc E such that $x \in E \cap (I \times I)$ and $F(E \cap (I \times I)) = \{c\}$. But $E \cap U \neq \phi$, since x is in the boundary of U. Hence, $U \cup E$ is connected and U was not maximal since E must also contain a point not in U since x is in the boundary of U. No point of B is in $F^{-1}(\{b\})$, for if $x \in B$ and $F(x) = b$, then since F is continuous at x, there is a disc G such that $x \in G \cap (I \times I)$ and $F(G \cap (I \times I)) = \{b\}$. But G contains no point of $F^{-1}(\{c\})$ and hence no point of U. Hence, $F(B) \subset \{a, d\}$. But $F(p) = a$ and $F(q) = d$. Hence, $F(B) = \{a, d\}$. Since $\{a, b, c\} \in T$, $F^{-1}(d)$ is closed. Similarly $F^{-1}(a)$ is closed. Since P is closed, $P \cap F^{-1}(a)$ and $P \cap F^{-1}(d)$ are closed. But $P \subset B$ containing p and q. Thus $P = (P \cap F^{-1}(a)) \cup (P \cap F^{-1}(d))$ which is a contradiction since P is connected and $P \cap F^{-1}(a)$ and $P \cap F^{-1}(d)$ are non-empty closed sets. Thus, no such continuous function F can exist and f and h are not homotopic modulo b. Hence $\Pi_1(X, b)$ has at least two elements and $N(X, b)$ cannot be isomorphic to $\Pi_1(X, b)$.

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