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A PERTURBATION METHOD FOR A SET OF QUASI-LINEAR OSCILLATORY SYSTEMS WITH VARIABLE NATURAL FREQUENCY

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1. Introduction

The perturbation method of K.B. [1] and H. [2] enables us to obtain the approximate oscillations of a set of a weakly non-linear oscillatory system with constant natural frequency. However, there exist some physical systems whose natural frequency has a slowly varying parameter, for example: the motion of a simple pendulum with variable length, the distribution of the energy released in a nuclear-powered reactor as a result of a power excursion [3] [4], electrical circuits containing parameters that are time varying [5], and so on.

In general the mathematical analysis of such systems leads to the second order diff erential equation of the form:

$$\ddot{x}+\omega^2(t)x=\varepsilon F(x,\ \dot{x}) \tag{1}$$

where $\omega^2(t)$ is an arbitrary function of the slowly varying parameter t, ε is a small parameter and F is an arbitrary function of the variable x, $\dot{x}$. The proposed procedure for analysing such systems is based on the extension of the method of K.B. or H. for treating the oscillatory system with constant frequency.

2. Proposed method

The homogeneous equation corresponding to equation (1) (for $\varepsilon=0$) is a linear diff erential equation with a slowly varying parameter, close to an exact integrable one when ω satisfies the condition $\frac{\dot{\omega}}{\omega^2}<<1$. An approximate solution is obtainable in the form of the constant ω solution [6].

$$\left.\begin{aligned} x(t)&=A\sin\psi(t)\\ \dot{x}(t)&=A\omega\cos\psi(t)\end{aligned}\right\} \tag{2}$$

where $\psi=\omega t+\phi$, A and ϕ are the integration constants. Equation (2) are used as a generating solution, when $\varepsilon\neq 0$, A and ϕ are considered unknown functions of t, and ω is a known function of t in equation (2). By differentiating the first equation of (2) and taking in our consideration the second equation of(2), we obtain the following

$$\dot{\omega}t\ A\cos\psi+A\dot{\phi}\cos\psi+A\ \sin\psi=0 \tag{3}$$

Also when differentiating the second equation of (2) and substituting into equation (1) we obtain

$$\dot{\omega}A\ \cos\psi-\omega\dot{\omega}t\ A\sin\psi-\omega\dot{\phi}\ \sin\psi+\dot{A}\omega\ \cos\psi=\varepsilon F \tag{4}$$

Solving (3) and (4) for $\dot{A}$ and $\dot{\phi}$ yields to

$$\dot{\phi}=-\omega t-\sin\psi\,\frac{(\varepsilon F-\dot{\omega}A\ \cos\psi)}{\omega A} \tag{5}$$

$$\dot{A}=\frac{\varepsilon F\ \cos\psi}{\omega}-\frac{\dot{\omega}A}{\omega}\cos^2\psi \tag{6}$$

The set of differential equations (5), (6) are equivalent to equation (1). For $\frac{\dot{\omega}}{\omega^2}<<1$, and when A and ϕ changes slowly with the independent variable t the amplitude A and the phase ϕ may be obtaind by taking the average of R.H.S of equation (5) and (6) over one period. Then

$$\left\langle\frac{dA}{dt}\right\rangle=\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi}F(A\ \sin\psi,\ A\omega\ \cos\psi)\ \cos\psi d\psi-\frac{\dot{\omega}A}{2\pi\omega}\int_0^{2\pi}\cos^2\psi d\psi \tag{7}$$

$$\left\langle\frac{d\psi}{dt}\right\rangle=\frac{-\varepsilon}{2\pi\omega A}\int_0^{2\pi}F(A\ \sin\psi,\ A\omega\ \cos\psi)\ \sin\psi\ d\psi+\omega \tag{8}$$

where $\langle\omega\rangle=\omega$.

In the above expressions A and ϕ are taken constants under the integral signs. Carrying the average one can obtain the amplitude, the phase and an analytical expression of the solution as a first approximation. It is interesting to note that the resulting solution obtained by the proposed method for $\varepsilon=0$ has the same form of the solution using the first BWK approximation [5]. Also, by taking ω as a constant we get the same result as the (K. B) and Haag methods.

For the purpose of comparison, it is interesting to justify the applicability of the proposed technique for different oscillatory systems with known solutions.

3. Applications

(1) Mathieu equation

In this case we have

$$\omega^2(t)=\omega_0^2(1-2h\ \cos\Omega t),\ \ \varepsilon=0$$

Applying (7) and (8) we get

$$\frac{dA}{dt}=-\frac{\dot{\omega}A}{2\omega}\ \text{implies,}\ \ A=\frac{c}{\sqrt{\omega}} \tag{9}$$

$$\frac{d\psi}{dt}=\omega \text{ implies, } \psi=\int\omega dt+\phi_0 \tag{10}$$

where c and ϕ_0 are arbitrary constants.

Substituting from (9) and (10) into (2) we get

$$x=\frac{c}{\omega_0\sqrt{1-2h\cos\Omega t}}\sin\left(\int\omega dt+\phi_0\right) \tag{11}$$

This result agrees with the familiar result obtained from BWK approximation method. The same result can be obtained using the perturbation method [7] by considering $\varepsilon=2h\omega_0^2$ as a small parameter.

(2) Damped Mathieu equation.

In this case $F(x, \dot{x})=-\dot{x}$, and $\omega(t)$ is defined as first application. By applying our approach we get from (7) and (8) the following

$$\frac{dA}{dt}=-\frac{A}{2}\left(\varepsilon+\frac{\dot{\omega}}{\omega}\right) \tag{12}$$

$$\frac{d\psi}{dt}=\omega$$

from which we deduce:

$$A=\frac{c}{\omega_0\sqrt{1-2h\cos\Omega t}}e^{-(\varepsilon/2)t},\quad \psi=\int\omega dt+\phi_0. \tag{13}$$

hence

$$x(\mathrm{t})=\frac{c}{\omega_0\sqrt{1-2h\cos\Omega t}}e^{-(\varepsilon/2)t}\sin\left(\int\omega dt+\phi_0\right) \tag{14}$$

which is the same result obtained by [5].

(3) The Van der Pol oscillatory type.

In this case we have: $F(x, \dot{x})=-(x^2-1)\dot{x}$

From equations (7) and (8) we get

$$\begin{aligned}\frac{dA}{dt}&=\frac{-\varepsilon}{2\pi}\int_0^{2\pi}(A^3\sin^2\psi\cos^2\psi-A\cos^2\psi)\,d\psi-\frac{\dot{\omega}}{2\pi\omega}\int_0^{2\pi}\cos^2\psi d\psi\\&=\frac{-\varepsilon A^3}{8}+\frac{\varepsilon A}{2}-\frac{\dot{\omega}}{2\omega}\end{aligned} \tag{15}$$

and

$$\frac{d\psi}{dt}=\frac{\varepsilon A^2}{2\pi}\int_0^{2\pi}\sin^3\psi\cos\psi d\psi-\frac{\varepsilon}{2}\int_0^{2\pi}\sin\psi\cos\psi d\psi+\omega=\omega \tag{16}$$

By solving equations (15) and (16) we have

$$\frac{1}{A^2}=\frac{\varepsilon e^{-\varepsilon t}}{4}\int\frac{e^{\varepsilon t}}{\omega}dt+\frac{ce^{-(\varepsilon/2)t}}{\omega} \tag{17}$$

$$\psi=\int\omega dt+\phi_0 \tag{18}$$

from which the solution may be obtained. In this case ω is defined as in [8]. If we take ω as a constant, the above result agrees with the result obtained by the K. B method.

(4) General Lord Rayleigh oscillatory type

In this application we have $F(x,\ \dot{x})=2k\dot{x}-c\dot{x}^3$ where k and c are positives. From (7) and (8) we have

$$\frac{dA}{dt}=kA-\frac{3}{8}cA^3\omega^2-\frac{\dot{\omega}}{2\omega}A \tag{19}$$

$$\frac{d\psi}{dt}=\omega \tag{20}$$

After the integration of equations (19) and (20) we can obtain

$$\frac{1}{A^2}=ce^{-kt}\omega\int e^{kt}\omega dt+A_0\omega e^{-kt} \tag{21}$$

$$\psi=\int\omega dt+\phi_0 \tag{22}$$

where A_0 and ϕ_0 are the integration constants.
From (21) and (22), the solution can be obtained where ω is defined and also the solution of the Lord Rayleigh oscillatry system with constant ω can be obtained.

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