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A NOTE ON CONTINUATION AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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This note is concerned with some properties of the solutions of the perturbed nonlinear second order differential equation

$$x''+f(x(t))g(x'(t))=h(t,\ x(t),\ x'(t))',=\frac{d}{dt} \tag{1}$$

where $f:R\to R$, $g:R\to R_+$, $h:I\times R^2\to R$ are continuous functions and $R=(-\infty,\infty)$, $R_+=(0,\ \infty)$, $I=[0,\ \infty)$.

We shall give sufficient conditions for all solutions of (1) to be continuable to the right of their initial value $t_0\in I$, and for all solutions $x(t)$ of (1) together with derivative $x'(t)$ to be bounded on I.

DEFINITIONS. (i) By continuable we mean a solution which is defined on a half-line $[t_0,\ \infty)$.

(ii) A solution $x(t)$ of (1) is said to be oscillatory if it has no last zero, otherwise it is nonoscillatory.

Let:

$$F(x)=\int_0^x f(s)\ ds\geq 0 \text{ for all } x\in R, \tag{2}$$

$$G(y)=\int_0^y [s/g(s)]\ ds,\ g(0)\neq 0,\ \lim_{|y|\to\infty} G(y)=\infty, \tag{3}$$

where $y(t)=x'(t)$

Our main assumptions are:

(i) There exists a continuous function $u:I\to R$ such that $|h(t,\ x,\ x')|\leq u(t)$,

(ii) There exists nonnegative constant M such that

$$|y|/g(y)\leq MG(y) \text{ for } |y|\geq 1. \tag{4}$$

THEOREM 1. *Under the conditions stated above, any solution $x(t)$ of equation (1) is continuable to the right of its initial t-value t_0.*

PROOF. Let $x(t)$ be a solution of equation (1) with initial t-value $t_0\in I$. Suppose, on the contrary, that $x(t)$ can not be continued past the finite point $T>t_0$, $T\in I$. It sufficies to show that $x(t)$ remains bounded as t approaches T

from the left.

Let: $V(t, x, x')=G(y)+F(x)+C$ where C is a nonnegative constant. Then

$$V'(t, x, x')=\frac{yh}{g(y)}\leq u[N+MG(y)]$$

Integrating both sides from t_0 to t and noting that $u(t)$ is bounded on $[t_0, T]$ we have

$$v(t)\leq C_1+M\int_{t_0}^{t} u(s)\ G(y(s))ds$$

Thus

$$G(y(t))\leq V<C_1+\int_{t_0}^{t} u(s)\ MG\ (y(s))ds.$$

Using Gronwall-Bellman inequality [3] there is a constant C_2 depending on $u(t)$ but not on $G(y)$ such that for all $t\ [t_0, T)$

$$G(y(t))\leq C_2<\infty$$

Thus $G(y(t))$ remains bounded as $t\to T$ from the left and so $y(t)=x'(t)$ remains bounded as $t\to T$ from the left. Consequently $x(t)$ is also bounded as $t\to T$. Thus we have a contradiction to our assumption that $x(t)$ is not continued past (finite) point T. This completes the proof.

COUNTEREXAMPLE TO THEOREM 1. Let $f\equiv 0$, $h(t, x, y)=2\phi(x)^3$ with $\phi(x)=\min(0, x)$ and $t_0=0$. Then $x=-1/(1-t)$ satisfies (1) on $[0, 1)$. One may take $u\equiv 0$.

In this section we will prove a boundedness theorem for solution $x(t)$ of equation (1) and its derivative $x'(t)$ by using a modification of the technique of the previous section. In addition to the given conditions we suppose that:

(i) $$\lim_{|x|\to\infty} F(x)=\infty, \qquad (5)$$

(ii) $$y^2/g(y)\leq MG(y)+N_1, \qquad (6)$$

$$|y|/g(y)\leq MG(y)+N_2, \text{ for all } y\in R.$$

These follow from condition (3).

(iii) There are continuous functions $r_i: I\to I$, $i=1, 2$ such that:

$$|h(t, x, y)|\leq r_1(t)+r_2(t)|y| \qquad (7)$$

for all $(t, x, y)\in I\times R^2$.

THEOREM 2. *Let assumptions* (2), (3), (6), (7) be *hold. Let* r_1 & r_2 *are integrable on* I. *Then for each solution* $x(t)$ *of equation* (1), *with initial* t*-value* $t_0\in I$, $x'(t)$ *is bounded on* $[t_0, \infty)$.

If in addition condition (5) *holds then* $x(t)$ *is bounded also on* $[t_0, \infty)$.

PROOF. Let $x(t)$ be a solution of equation (1) with initial t-value t_0, $t_0 \in I$. Multiplying (1) by $x'(t)/g(x(t))$ and integrating on $[t_0, t] \subset [t_0, T)$, we get:

$$G(y(t)) - G(y(t_0)) + F(x(t)) - F(x(t_0))$$
$$\leq \int_{t_0}^{t} h(s, x(s), y(s))\, y(s)/g(y(s))\, ds. \tag{8}$$

Using (6) & (7), inequality (8) can be written in the form

$$|G(y(t)) - G(y(t_0)) + F(x(t)) - F(x(t_0))| \leq \int_{t_0}^{t} \left| [r_1(s) + r_2(s) y(s)] \frac{y(s)}{g(s)} \right| ds,$$
$$\leq \int_{t_0}^{t} r_1(M\, G(y) + N_1) + r_2(M\, g(y) + N_2)\, ds,$$
$$\leq M \int_{t_0}^{t} [r_1(s) + r_2(s)]\, G(y)\, ds + \int_{t_0}^{t} (r_1(s) N_1 + r_2(s) N_2)\, ds,$$
$$\leq M \int_{t_0}^{t} (r_1(s) + r_2(s))\, G(y)\, ds + m(t), \tag{9}$$

where

$$m(t) = \int_{t_0}^{t} [N_1 r_1(s) + N_2 r_2(s)]\, ds.$$

Furthermore

$$m(t) \leq \int_{t_0}^{\infty} (N_1 r_1(s) + N_2 r_2(s))\, ds = m_0 < \infty. \tag{10}$$

Since $F(x) \to \infty$ as $|x| \to \infty$, $F(x)$ is bounded from below, say $F(x) \geq -K$. Let

$$V(t) = G(y(t)) + F(x(t)) + K. \tag{11}$$

Using (11), inequality (9) takes the form

$$V(t) \leq V(t_0) + m(t) + M \int_{t_0}^{t} [r_1(s) + r_2(s)]\, G(y(s))\, ds. \tag{12}$$

Hence

$$V(t) \leq V(t_0) + m_0 + M \int_{t_0}^{t} [r_1(s) + r_2(s)]\, V(s)\, ds,$$

by using (11) again.

By using Gronwall-Bellman inequality there is, then, a constant C, depending on $r_1(t)$ & $r_2(t)$ but not on $V(t)$ such that:

$$V(t) \leq (V(t_0) + m_0)\, C.$$

i.e. $G(y(t)) + F(x) \leq [G(y(t_0)) + F(x(t_0)) + m_0]\, C$.

It follow that $F(x(t))$ is bounded for $t \geq t_0$.

The conclusion of the theorem follows from (5)

COROLLARY. *If in addition to the hypotheses of theorem 2 and $G(y)\to\infty$ as $|y|\to\infty$, then all solutions $x(t)$ and the derivatives $x'(t)$ are bounded.*

PROOF. From the proof of theorem 2 we obtain

$$V(t)\leq(V(t_0)+m_0)\ C<\infty.$$

The boundedness of $y(t)$ then follows from the boundedness of $G(y(t))$ on $[t_0, T)$. An integration shows that $x(t)$ is also bounded on $[t_0, T)$. This completes the proof of the corollary.

THEOREM 3. *If $xf(x)>0$, for $x\neq 0$, $f'(x)\geq 0$,*

$$|h(t,\ x,\ x')|\leq u(t),\ \lim_{t\to\infty} u(t)=0,\ u\in L'(t_0,\ \infty), \tag{14}$$

$$g(y)\geq C>0, \tag{15}$$

and $x(t)$ is a bounded nonoscillatory solution of (1), then $\lim_{t\to\infty}\inf|x(t)|=0$.

PROOF. It will be convenient to consider the equivalent system

$$x'=y+\int_{t_0}^{t} h(s,\ x(s),\ y(s))\ ds, \tag{16}$$

$$y'=-f(x)\ g(y).$$

Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we can assume that $x(t)\neq 0$ on $[t_0,\ \infty)$. Let $x(t)>0$ for $t\geq t_1\geq t_0$. A similar arguments hold if $x(t)<0$ for $t\geq t_1>t_0$. On the contrary, suppose that $\lim_{t\to\infty}\inf\ x(t)\neq 0$. Then there exist positive numbers m *and* $t_2\geq t_1$ such that:

$$m<x(t)\ \text{for}\ t\geq t_2. \tag{17}$$

This condition together with (14) implies that

$$f(x(t))\geq A>0\ \text{for}\ t\geq t_2.$$

Thus from (16), by integration, we have

$$y(t)-y\ t_2=-\int_{t_2}^{t} f(x(s))\ g(y(s))\ ds\leq -AC(t-t_2). \tag{18}$$

Letting $t\to\infty$ in (18) we obtain

$$\lim_{t\to\infty} y(t)=-\infty.$$

Thus, for $t\geq t_3\geq t_2$ for some t_3 sufficiently large

$$x'(t)<0\ \text{for}\ t\geq t_3.$$

Then from (16), by integration, it follows that

$$x(t)\to-\infty\ \text{as}\ t\to\infty.$$

This, however, is a contradiction and hence

$$\lim_{t\to\infty}\inf\ x(t)=0.$$

This completes the proof of the theorem.

REMARKS. (1) It should be noted that x' is not necessarily bounded on $[t_0, \infty)$; this follows from the fact that the equation

$$x''+e^{2t}x=x'$$

has the bounded solution $x(t)=\sin(e^t)$ with an unbounded derivative.

(2) If, instead of the assumption (iii) before theorem 2, one assumes $|h(t, x, y)|\leq r_1(t)+r_2(t)|x|$ with r_1, r_2 integrable on I, the conclusion of theorem 2 is false, as example

$$x''+\frac{x}{1+x^2}=\frac{1}{1+t^2}\ x$$

which has $x=t$ as a solution.

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