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A NOTE ON BANACH*-ALGEBRA VALUED INNER PPRODUCT SPACES AND REPRESENTATIONS*

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1. Introduction

J.G. Bennet[2] considered vector spaces equipped with B^*-algebra-valued inner products. We investigated some properties of vector spaces equipped with Banach*-algebra valued inner products ([4], [10]). In this paper, we study the representations of a Banach*-algebra A on the A-valued inner product space. These results apply to the universal representations of the B^*-algebras. Also we obtain some relations among the linear maps from the A-valued pre-Hilbert space X into A with some different conditions.

2. Notations and preliminaries

A Banach algebra A with an involution * satisfying $\|a^*\|=\|a\|$ for all a in A will be called a *Banach*-algebra*. Throughout this paper, A denotes a Banach*-algebra with a multiplicative identity e and X denotes a complex linear space. A is *reduced* if $\{a\in A \mid f(a^*a)=0$ for all positive functionals f on $A\}=\{0\}$. B*-algebras are reduced. A is *symmetric* if the inverse $(e+a^*a)^{-1}$ exists in A for all $a\in A$. If A is reduced, then A is symmetric ([6] Corollary, p. 266, and II Proposition, p. 303). Let D be a *algebra. An element v in D is called the *quasi-unitary* if $vv^*=v^*v=v+v^*$. And D is a *U^*-algebra* if it is the linear span of its finite quasi-unitary elements. All Banach*-algebras are U^*-algebras([3]). Note that, if D has e, then $u\in D$ is unitary (i.e. $u^*u=uu^*=e$) if and only if e-u is a quasi-unitary, so in this case D is a U^*-algebra if and only if it is spanned by its unitaries.

Let B be a Banach*-algebra, D a *-algebra, and ϕ: $D\to B$ a linear map. We call ϕ *positive* if $\phi(b^*b)\geq 0$ for any $b\in D$. All algebras and linear spaces are those over the complex field C. X will be called an *A-valued inner product space* [4] if it is equipped with a map$\langle\cdot,\cdot\rangle$: $X\times X\to A$ such that

(i) $\langle\cdot,\cdot\rangle$ is linear in the first entry,

(ii) $\langle x,x\rangle\geq 0$ i.e., $\langle x,x\rangle$ is a positive element of A, for all x in X,

(iii) $\langle x,y\rangle=\langle y,x\rangle^*$, for all x,y in X.

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The map $\langle \cdot, \cdot \rangle$ is an A-valued inner product on X. Define

$$\|x\|_X = \sup \{f(\langle x, x\rangle)^{\frac{1}{2}},\ f \in \mathscr{P}_A\}$$

where $\mathscr{P}_A$ is the set of all positive linear functionals f on A such that $f(e) \leq 1$. Then $\|\cdot\|_X$ is a seminorm on X. Note that $\|x\|_* = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is not always a semi-norm on X. If A is a B^*-algebra, then $\|x\|_X = \|x\|_*$. An A-valued inner product space X will be called an *A-valued inner product module* if it is a right A-module satisfying $\langle xa, y\rangle = \langle x, y\rangle\, a$ for all x, y in X and a in A.

3. *-Representations of Banach*-algebras Induced by positive maps

DEFINITION 3.1. An A-valued inner product space (or module) X will be called an *A-valued pre-Hilbert space* (or *module*) if $\|\cdot\|_X$ is a norm on X.

An A-valued inner product space X is an *A-valued Hilbert space* if which is complete with respect to the norm $\|\cdot\|_X$.

In particular, if A is a B^*-algebra, then an A-valued inner product space (module) $(X, \|\cdot\|_X)$ is an A-valued pre-Hilbert space (module). If A is reduced and $\langle \cdot, \cdot \rangle$ is definite, then any A-valued inner product space (or module) is an A-valued pre-Hilbert space (or module). If A is not reduced, then the argument is not always true. If X is an A-valued inner product space, then it is easy to show that the following statements are equivalent:

(i) X is an A-valued pre-Hilbert space,

(ii) $\{x \mid f(\langle x, x\rangle) = 0 \text{ for all } f \in \mathscr{P}_A\} = \{0\}$,

(iii) $\{\langle x, x\rangle \mid f(\langle x, x\rangle) = 0 \text{ for all } f \in \mathscr{P}_A\} = \{0\}$ and $\langle \cdot, \cdot \rangle$ is definite.

Suppose that X is an A-valued pre-Hilbert space. Let $\mathscr{A}(X)$ be the family of $T \in B(X)$ for which there exists $T^* \in B(X)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x, y in X. Then $\mathscr{A}(X)$ is a normed *-algebra with $\|T^*T\| = \|T\|^2$ where $\|\,,\|$ denotes the operator norm on $B(X)$. Let X and Y be A-valued pre-Hilbert spaces and let B be a *-algebra. Suppose that $\pi : B \to \mathscr{A}(X)$ and $\varphi : B \to \mathscr{A}(Y)$ are *-representations of B on X and Y respectively, for which $\pi(B)\varepsilon_0 = X$ and $\varphi(B)\eta_0 = Y$ $(\varepsilon_0 \in X,\ \eta_0 \in Y)$. Let $\Phi : B \to A$ be a positive *-map such that $\Phi(a) = \langle \pi(a)\varepsilon_0, \varepsilon_0 \rangle$ and let $\Psi : B \to A$ be a positive *-map such that $\Psi(a) = \langle \varphi(a)\eta_0, \eta_0 \rangle$. Then we have:

LEMMA 3.2. *If $\Phi(a) = \varphi(a)$ for all a in a *-algebra B, then there exists an isometric isomorphism $V : X \to Y$ such that $\varphi(a) = V\pi(a)V^{-1}$ for all a in B.*

PROOF. Define $V : X \to Y$ by $V(\pi(a)\varepsilon_0) = \varphi(a)\eta_0$, for all a in B. Since $\Phi(a) =$

$\varphi(a)$ for all a in B,

$$\begin{aligned}\|\pi(a)\varepsilon_0\|_X^2 &= \sup_{f\in\mathscr{P}_A} f(<\pi(a)\varepsilon_0,\ \pi(a)\varepsilon_0>_X)\\ &= \sup_{f\in\mathscr{P}_A} f(<\pi(a^*)\pi(a)\varepsilon_0,\ \varepsilon_0>_X)\\ &= \sup_{f\in\mathscr{P}_A} f(<\pi(a^*a)\varepsilon_0,\ \varepsilon_0>_X)\\ &= \sup_{f\in\mathscr{P}_A} f(<\varphi(a^*a)\eta_0,\ \eta_0>_Y)\\ &= \sup_{f\in\mathscr{P}_A} f(<\varphi(a)\eta_0,\ \varphi(a)\eta_0>_Y)\\ &= \|\varphi(a)\eta_0\|_Y^2\ (a\in B)\end{aligned}$$

Clearly, V is an isometric surjective linear map. Hence V is a required map.

THEOREM 3.3. *Let B be a U^*-algebra and let A be a reduced Banach*-algebra.*

(1) *If $\Phi: B\to A$ is a positive map, then there exists an A-valued pre-Hilbert space X and a unital *-representation $\pi: B\to\mathscr{A}(X)$ and $\varepsilon_0\in X$ such that $\Phi(a)=<\pi(a)\varepsilon_0,\ \varepsilon_0>$ for all $a\in B$ and $\pi(B)\varepsilon_0=X$.*

(2) *The *-representation π is unique up to equivalence by Φ.*

(3) *If X is an A-valued pre-Hilbert space and if $\pi: B\to\mathscr{A}(X)$ is a *-representation with $\pi(B)\varepsilon_0=X$ for some $\varepsilon_0\in X$, then there exists a positive map $\Phi: B\to A$ such that $\Phi(a)=<\pi(a)\varepsilon_0,\ \varepsilon_0>$ for all a in B.*

PROOF. (1) Define $\langle x, y\rangle=\Phi(y^*x)$ for $x, y\in B$. Then $\langle\cdot,\cdot\rangle$ is an A-valued inner product on B. Let $N=\{x\in B| f(\langle x, x\rangle)=0$ for $f\in\mathscr{P}_A\}$. Then N is a linear subspace of B. Write $X=B/N$. We denote $\bar{x}=x+N$. Define $\langle\langle\bar{x}, \bar{y}\rangle\rangle=\langle x, y\rangle$, for all x, y in B. If $x_1+N=x_2+N$ and $y_1+N=y_2+N$, then $x_1-x_2\in N$, and $y_1-y_2\in N$. Since A is reduced and by the Cauchy Schwarz Inequality, $\langle x_1, y_1\rangle=\langle x_2, y_2\rangle$. Hence $\langle\langle\ ,\ \rangle\rangle$ is well defined. And $\langle\langle\ .,\ .\rangle\rangle$ is an A-valued inner product on X. Hence X is an A-valued pre-Hilbert space. For each $a\in B$, define a linear map $\pi(a): X\to X$ by $\pi(a)\bar{x}=\overline{ax}$. Let u be a unital element of B, then $\langle ux, ux\rangle=\langle x, x\rangle$ for all x in B. Since B is U^*-algebra, we have the linear map $\pi(a)$ is well defined, π is a linear homomorphism of B into $B(X)$ and

$$\|\pi(a)x\|_X\le M\|\bar{x}\|_X$$

for some positive real number $M>0$. Hence $\pi(a)\in B(X)$ for all $a\in B$. Since $\langle\langle(a)\bar{x}, \bar{y}\rangle\rangle=\Phi(y^*ax)=\Phi((a^*y)^*x)=\langle\langle\bar{x}, \pi(a^*)\bar{y}\rangle\rangle$, $\pi(a^*)=\pi(a)^*$ for all $a\in B$. Hence $\pi(a)\in\mathscr{A}(X)$. Also $\pi(e)\bar{x}=\overline{ex}=\bar{x}$. Thus π is a unital *-representation of B on X. Taking $\xi_0=\bar{e}=e+N$, we have $\pi(B)\bar{e}=X$.

(2) is obvious by Lemma 3.2.

(3) Define $\Phi: B\to A$ by $\Phi(a)=\langle\pi(a)\xi_0, \xi_0\rangle$ for all $a\in B$. Then we can check

that Φ is a required map.

REMARK 3.4. In the above theorem, if A is a B^*-algebra, then there exists an A-valued Hilbert space X and unital *-representation π of B on X and $\xi_0 \in X$ such that $\Phi(a) = \langle \pi(a)\xi_0, \xi_0 \rangle$ for all $a \in B$ and $\overline{\pi(B)\xi_0} = X$.

Clearly, if B is Banach*-algebra, then this representation π is continuous, $\|\pi\| \leq 1$ and Φ is continuous.

REMARK 3.5[1]. Let B be a U^*-algebra, if $\Phi : B \to \boldsymbol{C}$ is a positive functional, then there exists a Hilbert space X and unital *-representation $\pi : B \to B(X)$ and $\xi_0 \in X$ such that $\Phi(\mathrm{a}) = \langle \pi(\mathrm{a})\xi_0, \xi_0 \rangle$ for all $a \in B$.

By Theorem 3.3 and Remark 3.4, we have following Corollary:

COROLLARY 3.6[9]. *Let B be a U^*-algebra and let A be a B^*-algebra. If $\Phi : B \to A$ is a positive map.*

1) *If two *-representations are unitarily equivalent and one is cyclic, then so is the other.*

2) *A necessary and sufficent condition that two cyclic *-representations π and π' of B on X and X' respectively be equivalent is that there exist cyclic vectors $\xi \in X$ and $\xi' \in X'$ such that $\langle \pi(a)\xi, \xi \rangle = \langle \pi'(a)\xi', \xi' \rangle$ for any $a \in B$.*

4. Linear maps from X into A with some conditions

In this section, A denotes a symmetric Banach*-algebra with a unit e. We define a trivial A-valued inner product on A by $\langle a, b \rangle = b^*a$ for all $a, b \in A$. $\|a\|_A$ denotes the seminorm induced by $\langle \cdot, \cdot \rangle$. Let X be an A-valued inner product module. For a linear map $T : X \to A$, if we consider the following statements:

(1) $\|Tx\|_A \leq K_1 \|x\|_*$

(2) $\|Tx\| \leq K_2 \|x\|_*$

(3) $\|Tx\|_A \leq K_3 \|x\|_X$

(4) $\|Tx\| \leq K_4 \|x\|_X$ for any $x \in X$

(5) $T \in \hat{X}$, where $\hat{X}$ denotes the family of maps $\hat{x} : X \to A$ defined by $\hat{x}(y) = \langle y, x \rangle$ for all y in X, for some positive real numbers K_1, K_2, K_3, and $K_4 > 0$, then we have the following diagram:

$$\begin{array}{ccc} (5) \Longrightarrow & (3) & \Longleftarrow (4) \\ & \Big\Downarrow & \Big\Downarrow \\ & (1) & \Longleftarrow (2) \end{array}$$

The converse of each statement is not true, and we don't have relations between (3) and (2). For the cases of the converse of each statement, we give the following examples. In particular, when A is a B^*-algebra, (1), (2), (3), and (4) are equivalent.

EXAMPLES 4.1. Let A be the symmetric Banach*-algebra of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

with

$$\left\|\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}\right\| = |\alpha| + |\beta|, \quad \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}^* = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\alpha} \end{pmatrix}$$

for all α, $\beta \in C$. Suppose that $X = A$. Define $\langle a, b \rangle = b^* a$ for all a, $b \in X$. Then X is an A-valued inner product module. Considering the identity map i_X, we have (3), and hence (1). But we don't have (4) and (2). Suppose that $X = C$. Define

$$\lambda \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \lambda\alpha, \text{ and } \langle \alpha, \beta \rangle = \begin{pmatrix} 0 & \alpha\bar{\beta} \\ 0 & 0 \end{pmatrix}$$

for all α, β, $\lambda \in C$. Then X is an A-valued inner product module. Define $T : X \to A$ by

$$Tx = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

for all x in X. Then we have (2). But we don't have (4). Also, define $T : X \to A$ by

$$Tx = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

for all x in X. Then we have (1) and (2). But we don't have (3). A counter example for $(3) \not\Longrightarrow (5)$ is given in [7]. We donote that $X^r = \{T \mid T : X \longrightarrow A$ is a linear map with (3)$\}$. Then $\hat{X} \subset X^r$. If we define vector operations on $\mathscr{L}(X, A)$ by $(\lambda\tau)(x) = \bar{\lambda}\tau(x)$ and $(\tau + \tau')(x) = \tau(x) + \tau'(x)$ for all $\lambda \in C$, τ, $\tau' \in \mathscr{L}(X, A)$, $x \in X$, then $\mathscr{L}(X, A)$ is a linear space. Let $S(X^r)$ be the linear subspace generated by X^r. If we define $(\tau b)(x) = b^*\tau(x)$ for all b in A, x in X, then $S(X^r)$ is a right A-module.

DEFINITION 4.2. An A-valued pre-Hilbert space (module) X is called *strong* if $\|\langle x, y \rangle\| \le \|x\|_X \|y\|_X$, for all x, y in X.

If A is a B^*-algebra, then an A-valued pre-Hilbert space is strong. But the converse is not necessary true. If an A-valued pre-Hilbert space X is strong, we can check that the map $x \to \hat{x}$ is a isometric module map of X into $S(X^r)$. When B is a B^*-algebra, the following theorems are supported by [7]. We consider them on an A-valued pre-Hilbert modules.

THEOREM 4.3. *Let X be an A-valued pre-Hibert module with $S(X^r) = \hat{X}$ and Y be a strong A-valued pre-Hilbert module. If $T : X \to Y$ is a bounded linear module map, then there exists a bounded linear module map $T^* : Y \to X$ such that $\langle x, T^*y \rangle = \langle Tx, y \rangle$ for all $x \in X$, $y \in Y$.*

PROOF. Define $\phi_y : X \to A$ by $\phi_y(x) = \langle Tx, y \rangle$ for each fixed y in Y. Then

$$\begin{aligned}\phi_y(x)^*(\phi_y(x)) &= \langle y, Tx \rangle \langle Tx, y \rangle \\ &\leq \|\langle y, y \rangle\| \langle Tx, Tx \rangle.\end{aligned}$$

Hence $\|\phi_y(x)\|_A^2 \leq K\|x\|_X^2$ for some real $K > 0$, and $\phi_y \in X^r \subset S(X^r)$. Since $S(X^r) = \hat{X}$, there exists a unique element $z \in X$ such that $\phi_y(x) = \langle x, z \rangle$ for all $x \in X$. Define $T^* : Y \to X$ by $T^*y = z$. Then $\langle x, T^*y \rangle = \phi_y(x) = \langle Tx, y \rangle$ for all $x \in X$, $y \in Y$. There exists a positive real number K such that $\|\langle x, T^*y\| \leq K\|x\|_X \|y\|_Y$ for all $x \in X$, $y \in Y$. It is easy to check that T^* is a required map.

COROLLARY 4.4. *If X is a strong A-valued pre-Hilbert module with $S(X^r) = \hat{X}$, then every module map in $B(X)$ belongs to $\mathscr{A}(X)$.*

In particular, if B is B^*-algebra, in view of 4.3 and Corollary 4.4, we have shown;

COROLLARY 4.5. ([7]) *Let X be a self-dual Hilbert B-module, Y a pre-Hilbert B-module, and $T : X \to Y$ a bounded module map. Then there is a bounded module map $T^* : Y \to X$ such that $\langle x, T^*y \rangle = \langle Tx, y \rangle$ for any x in X and y in Y.*

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