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ENVELOPING SERIES FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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1. Introduction

In recent years there has developed a growing interest in the study of approximation theory as applied to certain of the Special Functions of Mathematical Physics, see Luke [4]. Of special interest is the determination of upper and lower bound estimates for such functions whenever these can be found. The purpose of this work is to find necessary and sufficient conditions under which the Maclaurin series expansion for any of the hypergeometric functions of one variable is an upper and/or lower bound. The starting point of this investigation is an inequality of Bernoulli which was extended by Gerber [2] and an elementary proof of which was obtained by Ross [10], namely

$$(-\alpha)_{n+1}\ (-x)^{n+1}\left[(1+x)^{\alpha}-\sum_{\nu=0}^{n}\frac{(-\alpha)_{\nu}(-x)^{\nu}}{\nu!}\right]\geq 0,\ \text{for } x>-1\ \alpha\in\boldsymbol{R},$$
$$n=0,1,2,\cdots \tag{1}$$

where $(a)_{r}=\Gamma(a+\gamma)/\Gamma(a)$ is the Pochhammer symbol and the term in square brackets is zero for $x\neq 0$ and for $n>0$, if and only if, α is one of the integers $0,1,2\ldots n$. Note that the above result remains valid even if the series is not convergent. The current analysis is particularly important for it leads in quite a simple way to the development of enveloping series for some of the special functions of mathematical physics, see Pólya and Szegö [9], to an estimation of their smallest zeros, see Whittaker and Watson [13] as well as the book of Luke [6].

In this paper, it is found convenient to define the generalized and lower order hypergeometric functions by the series

$${}_pF_q(x)\equiv{}_pF_q[a_1,\cdots a_p\,;\,b_1,\cdots b_q\,;\,x]=\sum_{\nu=0}^{\infty}\frac{(a_1)_{\nu}\cdots(a_p)_{\nu}x^{\nu}}{(b_1)_{\nu}\cdots(b_q)_{\nu}\nu!} \tag{2}$$

where the a_i, $i=1,2,\cdots,p$ and the b_j, $j=1,2,\ldots q$ are regarded as parameters in the numerator and denominator respectively and x is the independent variable.

Here the Pochhammer symbol $(c)_\nu$ stands for the product of ν terms so that

$$(c)_\nu = c(c+1)(c+2)\dots(c+\nu-1) \text{ when } \nu=1,2,3,\dots \text{ and } (c)_0=1.$$

In case p and/or $q=0$ the parameters in the numerator and/or denominator are simply omitted. Since the series referred to in (2) is often cumbersome to write down in more complicated formulae the following alternative notations, due to Slater [12] will be used, namely

$${}_pF_q(x) \equiv {}_pF_q\ [(a_p)\ ;\ (b_q)\ ;\ x] \equiv \sum_{\nu=0}^{\infty} \frac{((a_p))_\nu x^\nu}{((b_q))_\nu \nu!}$$

whenever the series is meaningful and converges for some $x \neq 0$. Here (c_k) denotes the sequence of parameters $c_1, c_2, \dots, c_k$ and $((c_k))_\nu$ denotes $\prod_{i=1}^{k} (c_i)_\nu$.

It is assumed throughout the paper that, if any one of the b_j's is a negative integer or zero then the series terminates at a stage before the zero divisor appears. Thus, if $-N$ is the largest of the non-positive integers in the set $b_1, b_2, \dots, b_q$ then at least one of the parameters $a_1, a_2, \dots, a_p$ must be a non-positive integer greater than or equal to $-N$. Note that the convention is adopted that the series terminates at $\nu=M$ if $-a_l=-b_k=M$ and M is the smallest positive integer with this property. It follows that the M^{th} partial sum of the series ${}_pF_q(x)$ can be written in the form

$$\sum_{\nu=0}^{\infty} \frac{(-M)_\nu ((a_p))_\nu x^\nu}{(-M)_\nu ((b_q))_\nu \nu!}.$$

Apart from the above possibility the series is convergent

(a) for all complex x, when $p \leq q$,

(b) for $|x|<1$, when $p=q+1$,

(c) for $x=1$ provided that $\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i > 0$ when $p=q+1$,

(d) for $x=-1$ provided that $\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i > -1$ when $p=q+1$,

and (e) $x=0$ whenever $p>q+1$.

On the other hand, it is easy to show that for all other values of x, p and q the series in (2) is divergent, unless it terminates. Throughout this paper the convention will be adopted that no b_j is equal to a_i since this merely has the effect of reducing the order of the hypergeometric function. The situation where one or more of the parameters a_i is zero will be ignored as well for then the hypergeometric series contains only one term.

2. The main theorems

THEOREM A. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two sequences, possibly null sequences, of real numbers then*

$$ {}_pF_q[(a_p);(b_q);x]=\frac{\Gamma(b_q)}{\Gamma(a_p)\Gamma(b_q-a_p)}\int_0^1 t^{a_p-1}(1-t)^{b_q-a_p-1}\,{}_{p-1}F_{q-1}[(a_{p-1});(b_{q-1});xt]dt \tag{3} $$

provided that $b_q>a_p>0$.

This is easily proved by expanding the hypergeometric function in the integrand in powers of x and integrating the uniformly convergent series term-by-term, see Slater [12].

To begin, it is important to look at the case where $\{a_i\}$ and $\{b_j\}$ are null sequences.

LEMMA. *The exponential function* $e^x\equiv {}_0F_0[-;-;x]$ *satisfies*

$$ \frac{x^{n+1}}{(n+1)!}\left[{}_0F_0[-;-;x]-\sum_{\nu=0}^{n}\frac{x^\nu}{\nu!}\right]\geq 0, \text{ for all } x\in\boldsymbol{R}. \tag{4} $$

PROOF. Let $Y_n(x)=1-e^{-x}\sum_{\nu=0}^{n}\frac{x^\nu}{\nu!}$, then $Y_n(0)=0$

and $\frac{x^n}{(n+1)!}\frac{d}{dx}[Y_n(x)]=e^{-x}\frac{x^{2n}}{n!(n+1)!}\geq 0$, for all $x\in\boldsymbol{R}$.

See for comparison a similar inequality given in Pólya and Szegö [9].

THEOREM 1. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two, possibly null, sequences of non-negative real numbers then*

$$ x^{n+1}\left[{}_pF_p[(a_p);(b_p);x]-\sum_{\nu=0}^{n}\frac{((a_p))_\nu x^\nu}{((b_p))_\nu \nu!}\right]\geq 0, \tag{5} $$

for all $x\in\boldsymbol{R}$, *and* $n=1,2,3,\dots$.

PROOF. The preceeding Lemma shows that (5) is valid for $p=0$. When $p\neq 0$ and $((a_p))_1=0$ inequality (5) is an identity and there is no more to say. On the other hand, when $p\neq 0$ and the two sequences are such that $b_i>a_i>0$ for each $i=1,2,\dots,p$ then it is admissible to apply the representation Theorem A, p-times to the inequality (4) of the Lemma so obtaining the result (5).

The next step is to show that the inequality restrictions on the parameters a_i and b_j can be eased. This can be achieved by noting that there are no such restrictions for a ${}_0F_0(x)$ and then to use an inductive proof based on the above result. To begin, suppose that inequality in (5) is valid with a_i, $b_j>0$ for each ${}_mF_m(x)$ with $m=0,1,\dots,p-1$. Then by applying the relation (3) of Theorem A, it is seen that the function ${}_pF_p(x)$ satisfies (5) whenever $b_p>a_p$ and where $a_p>0$.

Now it is well known that ${}_pF_p$'s satisfy a simple three-term recurrence relation

$$a_{p\,p}F_q[(a_p)\,;\,(b_q)\,;\,x]+\frac{((a_p))_1 x}{((b_q))_1}\,{}_pF_q[(1+a_p)\,;\,(1+b_q)\,;\,x]$$
$$=a_{p\,p}F_q[(a_{p-1}),\ 1+a_p\,;\,(b_q)\,;\,x]$$

See Luke [4, p160]. This can be used to infer that their remainders satisfy a similar type of relation. In particular, on equating coefficients of x^n, it is found that

$$a_p\sum_{\nu=n+1}^{\infty}\frac{((a_p))_\nu x^\nu}{((b_p))_\nu \nu!}+\frac{((a_p))_1 x}{((b_p))_1}\sum_{\nu=n}^{\infty}\frac{((1+a_p))_\nu x^\nu}{((1+b_p))_\nu \nu!}$$
$$=a_p\sum_{\nu=n+1}^{\infty}\frac{(1+a_p)_\nu((a_{p-1}))_\nu x^\nu}{(b_p)_\nu((b_{p-1}))_\nu \nu!} \tag{6}$$

for $n=0,1,2,\dots$ and provided that each series is meaningful (by which is meant that no b_j is a negative integer and that the series converges for some $x\neq 0$, or else it terminates or terminated).

Now from (5) and the inductive hypothesis it can be deduced that

$$a_p x^{n+1}\sum_{\nu=n+1}^{\infty}\frac{((a_p))_\nu x^\nu}{((b_p))_\nu \nu!}\geq 0,$$

and

$$\frac{((a_p))_1}{((b_p))_1}x^{n+2}\sum_{\nu=n}^{\infty}\frac{((1+a_p))_\nu x^\nu}{((1+b_p))_\nu \nu!}\geq 0,\ \ n=1,2,3,\dots$$

for $b_p>a_p$ where $a_p>0$. Hence it follows, after multiplying (6) by x^{n+1} that

$$x^{n+1}\sum_{\nu=n+1}^{\infty}\frac{(1+a_p)_\nu((a_{p-1}))_\nu x^\nu}{(b_p)_\nu((b_{p-1}))_\nu \nu!}\geq 0,\ \text{for all } n=1,2,3,\dots$$

Hence, on using (6) a number of times it appears that the inequality restriction $b_p>a_p$ is no longer needed to establish (5) as long as each a_i and b_j is non-negative.

THEOREM 2. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two, with* $\{a_i\}$ *possibly null, sequences of non-negative real numbers then, for* $p<q$.

$$x^{n+1}[{}_pF_q[(a_p)\ ;\ (b_q)\ ;\ x]-\sum_{\nu=0}^{n}\frac{((a_p))_\nu x^\nu}{((b_p))_\nu \nu!}]\geq 0,$$

for all $x\in\boldsymbol{R}$, *and* $n=1,2,3,\ldots$

PROOF. This is based on a repeated application of a principle of confluence as applied to the particular hypergeometric function ${}_pF_p(x)$ discussed in the previous theorem, namely

$$\lim_{a_p\to\infty}{}_pF_q[(a_{p-1}),\ a_p\ ;\ (b_q)\ ;\ \frac{x}{a_p}]={}_{p-1}F_q[(a_{p-1})\ ;\ (b_q)\ ;\ x]$$

for $p=1,2,\ldots,q$ and all $x\in\boldsymbol{R}$, see Luke [4, p180].

The next step is to look at theorems which apply in case $p>q$.

THEOREM 3. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two, with* $\{b_j\}$ *possibly null, sequences with* $a_i,\ b_j>0$ *for* $i,\ j=1,2,\ldots,p,\ a_{p+1}\in\boldsymbol{R}$ *and* $-1<x<1$, *then*

$$(a_{p+1})_{n+1}x^{n+1}[{}_{p+1}F_p[(a_p),\ a_{p+1}\ ;\ (b_p)\ ;\ x]-\sum_{\nu=0}^{n}\frac{((a_{p+1}))_\nu x^\nu}{((b_p))_\nu \nu!}\geq 0, \tag{7}$$

for $n=1,2,3,\ldots$

PROOF. From the enveloping property of the Binomial expansion (1) it follows that

$$(a_{p+1})_{n+1}x^{n+1}[{}_1F_0[a_{p+1}\ ;\ -\ ;\ x]-\sum_{\nu=0}^{n}\frac{(a_{p+1})_\nu x^\nu}{\nu!}\geq 0,$$

for $x<1$ and any $a_{p+1}\in\boldsymbol{R}$. On applying the integral transformation referred to as equation (3) of Theorem A. p-times in succession and then the repeated application of (6) a suitable number of times the inequality in (7) is proven. The further restriction on x arises from a convergence condition which has to be placed on the integrals in question.

THEOREM 4. *Let* $\{a_i\}$ *and* $\{b_j\}$ *be two, possibly null, sequences of real numbers, then*

$$\frac{((a_p))_{n+1}}{((b_q))_{n+1}}\,x^{n+1}[{}_pF_q[(a_p)\ ;\ (b_q)\ ;\ x]-\sum_{\nu=0}^{n}\frac{((a_p))_\nu x^\nu}{((b_q))_\nu \nu!}\geq 0, \tag{8}$$

for $n=(m+1),\ (m+2),\ldots,$ *where* m *is the smallest positive integer for which each term of the sequences* $a_i+m+1,\ b_j+m+1$ *is positive and* $b_j\neq 0,-1,-2,\ldots$ *and*

(*i*) *for all real* x, *when* $p\leq q$

(*ii*) *for* $-1<x<1$ *when* $p=q+1$.

(*iii*) *for all* x *if* $p>q+1$ *and/or the series terminates since then the term in the* *large squared brackets is identically zero.*

In case $p=q+1$ with $-1<x<1$ the Theorem may apply with a smaller value of m if there exists an $m=M$ which is such that each term in the sequence $\{b_j+M+1\}$ is positive and all but one term in the sequence $\{a_i+M+1\}$ is positive.

PROOF. The case when a_i and b_j are all positive is contained in the earlier part of this paper. Hence, it is sufficient to assume that either one or more of the a_i's or the b_j's is negative. Clearly it is always possible to choose a positive integer m such that all members of the sequences b_j+m+1 and a_i+m+1 are positive. In that case Theorems can be used to show that

$$\frac{((a_p+m+1))_{n+1}}{((b_q+m+1))_{n+1}}\, x^{n+1}[{}_pF_q[(a_p+m+1)\,;\,(b_q+m+1)\,;\,x] - \sum_{\nu=0}^{n}\frac{((a_p+m+1))_\nu x^\nu}{((b_q+m+1))_\nu \nu!}]\geq 0 \tag{9}$$

under the conditions given in the statement of the Theorem.

Now on applying the transformation

$$\frac{((a_p+m))_1}{((b_q+m))_1}\int_0^x {}_pF_q[(a_p+m+1)\,;\,(b_q+m+1)\,;\,t]dt = {}_pF_q[(a_p+m)\,;\,(b_q+m)\,;\,x]-1$$

to the inequality (9), $(m+1)$ times the result follows.

The second part of the Theorem is an immediate consequence of Theorem 3.

It is worth noting that Theorem 4 cannot, in general, be strengthen in the sense that (8) does not always remain true for values of $n\leq m$. This is easily proved by looking at a particular example. Thus

$${}_2F_1(-1.5,1\,;\,-3.5\,;\,x)=1+\frac{3x}{7}+\frac{3x^2}{35}-\frac{x^3}{35}+\frac{3x^4}{35}+\frac{3x^5}{7}+\cdots \text{ for } |x|<1$$

and, if the theorem were applicable for $n=2$, this would imply that

$$\frac{x^3}{35}\geq\frac{3x^4}{35}+\frac{3x^5}{7} \text{ for some } x \text{ in } 0\leq x\leq 1$$

a result which is false when $x=0.2$.

3. Examples

The theorems in the previous section may be used to verify the one-sided approximations to Bessel functions $J_\alpha(x)$ and to Jacobi polynomials $P_n^{(\alpha,\,\beta)}(x)$ as referred to by Askey [1]. For, since

$$J_\alpha(x)=\frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1[-\,;\,\alpha+1\,;\,-x^2/4],\ \text{for } \alpha>-1 \text{ and } x\geq 0,$$

a simple application of Theorem 2 implies that

$$(-1)^{n+1}\Big[J_{\alpha}(x)-\sum_{\nu=0}^{n}\frac{(-1)^{\nu}(x/2)^{2\nu+\alpha}}{\Gamma(\nu+\alpha+1)\nu!}\Big]\geq 0,\ \text{for}\ \alpha>-1,\ x\geq 0$$

with $n=1,2,3,\ldots$. Observe that $(x/2)^{\alpha}$ and $J_{\alpha}(x)$ are underlined for $x=0$ if $\alpha<0$.

$$P_{m}^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_{m}}{m!}\ {}_{2}F_{1}[-m,\ m+\alpha+\beta+1\ ;\ \alpha+1\ ;\ \tfrac{1}{2}(1-x)]$$

for $\alpha,\beta,\ x\in\boldsymbol{R}$ and $m=0,1,2,\ldots$, an applcation of Theorem 3 leads to

$$(-1)^{n+1}\Big[P_{m}^{(\alpha,\beta)}(x)-\frac{(\alpha+1)\ _{m}}{m!}\sum_{\nu=0}^{n}\frac{(-m)_{\nu}(m+\alpha+\beta+1)_{\nu}(1-x)^{\nu}}{2^{\nu}(\alpha+1)_{\nu}\nu!}\Big]\geq 0,$$

for $\alpha+\beta>-1,\ \alpha>-1,\ -1<x<1$ and $n=1,2,3,\ldots m$.

Some interesting extensions of these inequalities are obtainable from identities relating to a product of hypergeometric functions. Thus it is known that

$$J_{\alpha}(x)J_{\beta}(x)=\frac{(x/2)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\ {}_{2}F_{3}\Big[\frac{\alpha+\beta+2}{2},\ \frac{\alpha+\beta+1}{2}\ ;\ \alpha+1\ ;\ \beta+1,\ \alpha+\beta+1\ ;\ -x^{2}\Big]$$

for $\alpha,\ \beta>-1$ and $x\geq 0$, see Luke [4, p314].

Hence, from Theorem 2 it follows that

$$(-1)^{n+1}\Big[J_{\alpha}(x)J_{\beta}(x)-\sum_{\nu=0}^{n}\frac{\left(\frac{\alpha+\beta+2}{2}\right)_{\nu}\left(\frac{\alpha+\beta+1}{2}\right)_{\nu}(-1)^{\nu}x^{2\nu+\alpha+\beta}}{2^{\alpha+\beta}\Gamma(\alpha+1+\nu)\Gamma(\beta+1+\nu)(\alpha+\beta+1)_{\nu}\nu!}\Big]\geq 0,$$

under the same conditions as stated above with $n=1,2,3,\ldots$.

Again the product of two Jacobi polynomials

$$P_{m}^{(\alpha,\beta)}(x)\ P_{m}^{(\beta,\alpha)}(x)=\frac{(\alpha+1)_{m}(\beta+1)_{m}}{m!\ m!}\ {}_{4}F_{3}[-m,\ \tfrac{1}{2}(\alpha+\beta+1),\ \tfrac{1}{2}(\alpha+\beta+2),\ m+\alpha+\beta+1\ ;\ \alpha+1,\ \beta+1,\ \alpha+\beta+1\ ;\ 1-x^{2}],$$

for $\alpha,\beta,x\in\boldsymbol{R}$ and $m=0,1,2,\ldots$, see Koornwinder [3]. Hence, from Theorem 3, it is seen that

$$(-1)^{+1}\Big[P_{m}^{(\alpha,\beta)}(x)\ P_{m}^{(\beta,\alpha)}(x)-\frac{(\alpha+1)_{m}(\beta+1)_{m}}{m!\ m!}\sum_{\nu=0}^{n}\frac{(-m)_{\nu}\left(\frac{\alpha+\beta+1}{2}\right)_{\nu}\left(\frac{\alpha+\beta+2}{2}\right)_{\nu}(m+\alpha+\beta+1)\nu(1-x^{2})^{\nu}}{(\alpha+1)_{\nu}(\beta+1)_{\nu}(\alpha+\beta+1)_{\nu}\nu!}\Big]\geq 0$$

for $\alpha,\beta>-1,\ -\sqrt{2}<x<\sqrt{2}$ and $n=1,2,3,\ldots,m$. Observe that this inequality is false when $n=0$ and $1<x<\sqrt{2}$.

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