Kyungpook Math. J.
Volume 25, Number 1
June, 1985

USE OF HANKEL TRANSFORM FOR INTEGRATION OF GAUSS'S HYPERGEOMETERIC SERIES

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The object of this paper is to apply Hankel Transformations simultaneously to the evaluation of integrals that involve Gauss' Hypergeometric series along with Bessel functions. This form of hypergeometric function can be obtained from Meijer's G-function by specializing parameters. Many of the integrals in this paper are not obtainable from generalized integrals (especially in neat form) appeared in research journals recently (see [2] to [24]) involving Meijer's G-function. In the present paper, the author has developed a interesting theorem to evaluate that kind of integrals, which are much more usable than the generalized integrals. The well-known Hankel Transform [1] of order v is defined by the following equation:

$$H_v[g(x):y]=\phi(y)=\int_0^\infty \sqrt{xy}\; J_v(xy)g(x)\,dx. \tag{1.1}$$

THEOREM 1. *Let*

(*i*) $\phi(y)$ *be the Hankel transform of* $f(x)$ *of order* v,

(*ii*) $K(p)$ *be the Hankel transform of* $x^{2(\alpha-v-1)}f(x)$ *of order* v, *then*

$$p^{2(\alpha-v-1)}K(p)=\frac{2^{2\alpha-2v-1}\Gamma(\alpha)}{\Gamma(v+1)\Gamma(v-\alpha+1)}\int_0^\infty \frac{y^{v+\frac{1}{2}}}{(1+y)^{2\alpha}}\,{}_2F_1\Big(\alpha,\ v+\frac{1}{2};\ 2v+1;\ \frac{4y}{(1+y)^2}\Big)\,\phi(y)\,dy \tag{2.1}$$

provided $f(x)$, $x^{2(\alpha-v-1)}f(x)$ *are continuous and absolutely integrable in* $(0,\ y)$ *or in* $(0,\ \infty)$, $^{*}R(v)>-\frac{1}{2}$, $R(2\alpha-v)>\frac{3}{2}$.

PROOF. We have, from (i)

$$\phi(y)=\int_0^\infty \sqrt{xy}\; J_v(xy)f(x)\,dx$$

$$\text{or } \phi(py)=\int_0^\infty \sqrt{pyx}\; J_v(pyx)f(x)\,dx. \tag{2.2}$$

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* These conditions can be relaxed.

Now multiplying both sides of (2.2) by

$y^{v+\frac{1}{2}}(1+y)^{-2\alpha}\ {}_2F_1(\alpha,\ v+\frac{1}{2};\ 2v+1;\ 4y(1+y)^{-2})$, *and integrating between the limits* (0, ∞), *we have*

$$(2.3)\quad \int_0^\infty \frac{y^{v+\frac{1}{2}}}{(1+y)^{2\alpha}}\ {}_2F_1(\alpha,\ v+\tfrac{1}{2};\ 2v+1;\ \frac{4y}{(1+y)^2})\phi(py)\,dy$$
$$=\int_0^\infty \frac{y^{v+\frac{1}{2}}}{(1+y)^{2\alpha}}\ {}_2F_1(\alpha,\ v+\tfrac{1}{2};\ 2v+1;\ \frac{4y}{(1+y)^2})\,dy$$
$$\int_0^\infty \sqrt{pxy}\ J_v(pyx)f(x)\,dx$$

On changing the order of integration on the right, which is justifiable due to the conditions given in the theorem, we have

$$\text{Right hand side}=\sqrt{p}\int_0^\infty \sqrt{x}f(x)\,dx\int_0^\infty \frac{y^{v+1}}{(1+y)^{2\alpha}}\ {}_2F_1(\alpha,\ v+\tfrac{1}{2},\ 2v+1;$$
$$\frac{4y}{(1+y)^2})\cdot J_v(pyx)\,dy,$$

$$(2.4)\quad =\frac{\Gamma(v+1)\ \Gamma(v-\alpha+1)}{\Gamma(\alpha)2^{2\alpha-2v-1}p^{2v-2\alpha+3/2}}\int_0^\infty x^{2\alpha-2v-3/2}\ J_v(px)f(x)\,dx.$$

(see [1], result 10, page 82)

But $K(p)$ is the Hankel transform of $x^{2(\alpha-v-1)}f(x)$ of order v. Hence we get

$$p^{2(\alpha-v-1)}K(p)=\frac{2^{2\alpha-2v-1}\Gamma(\alpha)}{\Gamma(v+1)\ \Gamma(v-\alpha+1)}\int_0^\infty \frac{y^{v+\frac{1}{2}}}{(1+y)^{2\alpha}}\ {}_2F_1(\alpha,\ v+\tfrac{1}{2};\ 2v+1;$$
$$\frac{4y}{(1+y)^2})\ \phi(py)\,dy.$$

1. **Applications.** Let $x^{2(\alpha-v-1)}f(x)=x^{-\frac{1}{2}}e^{-ax}$. The Hankel Transform of $x^{-1/2}e^{-ax}$ is

$$p^{1/2-v}(a^2+p^2)^{-\frac{1}{2}}\ [\sqrt{a^2+p^2}-a]^v\equiv K(p)\quad \text{(see [1], page 28, result 6)}$$

Hence from the theorem (result (2.1)), we get

$$(3.1)\quad \int_0^\infty \frac{y^{v+1}}{(1+y)^{2\alpha}(a^2+p^2y^2)^{v-\alpha+3/2}}\ P^{-v}_{2v-2\alpha+2}\left(\frac{a}{\sqrt{a^2+p^2y^2}}\right)\ {}_2F_1$$
$$\left[\begin{matrix}\alpha,v+\frac{1}{2};\\ 2v+1;\end{matrix}\ \frac{4y}{(1+y)^2}\right]dy=\frac{\Gamma(v+1)\Gamma(v-\alpha+1)2^{2v-2\alpha+1}}{\Gamma(\alpha)\ \Gamma(3v-2\alpha+3)}\cdot\frac{[\sqrt{a^2+p^2}-a]^v}{(a^2+p^2)^{1/2}p^{3v-2\alpha+2}},$$
$$R(\alpha)>0,\ R(v)>-1.$$

Table of integrals based on theorem 1.

No.	$x^{2(\alpha-v-1)}\ f(x)$	Integrals obtained from (2.1) by using known results (see [1], pp. 22~92).
3.2	$\dfrac{x^{v+\frac{1}{2}}}{(x^2+a^2)^{\mu+1}}$	$\int_0^\infty y^{v+\frac{1}{2}}(1+y)^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha, v+\frac{1}{2};\\ 2v+1;\end{matrix}\ \frac{4y}{(1+y)^2}\right]\Big\{\frac{\Gamma(2v-\alpha+2)\Gamma(\mu+\alpha-2v-1)(py)^{v+\frac{1}{2}}}{2^{v+1}\Gamma(\mu+1)\Gamma(v+1)a^{2(\mu-2v+\alpha-1)}}$ $\times {}_1F_2\left[\begin{matrix}2v-\alpha+2;\\ 2v-\alpha-\mu+2,\ v+1;\end{matrix}\ \frac{p^2y^2a^2}{4}\right]+\frac{\Gamma(2v-\alpha-\mu+1)(py)^{2\mu-3v+2\alpha-3/2}}{\Gamma(\alpha+\mu-v)2^{2\mu-3v+2\alpha-1}}$ $\times {}_1F_2\left[\begin{matrix}\mu+1;\\ \mu+\alpha-v,\ v+\alpha-2v;\end{matrix}\ \frac{p^2y^2a^2}{4}\right]\Big\}=\frac{\Gamma(v+1)\Gamma(v-\alpha+1)a^{v-\mu}p^{\mu-3/2+2\alpha-2\mu}}{\Gamma(\alpha)\Gamma(\mu+1)2^{2\alpha-2v+\mu-1}}\ K_{v-\mu}(ap)$, $R(2\alpha-v)>\frac{3}{2}$, $R(\alpha)>0$, $R(v)>-1$, $R(\mu-v+\alpha)>\frac{3}{4}+\frac{1}{2}R(v)$, $R(2\alpha-4v)<4$.
3.3	$x^{v-\frac{1}{2}}e^{-ax}$	$\int_0^\infty \frac{y^{2v+1}}{(1+y)^{2\alpha}(a^2+p^2y^2)^{v-\alpha+3/2}}\ {}_2F_1\left[\begin{matrix}\alpha, v+1/2;\\ 2v+1;\end{matrix}\ \frac{4y}{(1+y)^2}\right]\ {}_2F_1\left[\begin{matrix}2v-\alpha+3/2,\ \alpha-v-1;\\ v+1;\end{matrix}\ \frac{p^2y^2}{a^2+p^2y^2}\right]dy$ $=\frac{[\Gamma(v+1)]^2\Gamma(v-\alpha+1)\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(4v-2\alpha+3)\Gamma(\alpha)2^{(2\alpha-4v-1)}}\cdot\frac{p^{2(\alpha-v-1)}}{(a^2+p^2)^{v+\frac{1}{2}}}$, $R(\alpha)>0$, $R(v)>-\frac{1}{2}$, $R(4v-2\alpha+3)>0$.
3.4	$x^{v+\frac{1}{2}}e^{-ax^2}$	$\int_0^\infty y^v(1+y)^{-2\alpha}\exp\left(-\frac{y^2p^2}{8a}\right)M_{\frac{3v+3-\alpha}{2},\ \frac{v}{2}}\left(\frac{p^2y^2}{4a}\right)\ {}_2F_1\left(\alpha,\ v+\frac{1}{2};\ 2v+1;\ \frac{4h}{(1+y)^2}\right)dy$ $=\frac{[\Gamma(v+1)]^2\ \Gamma(v-\alpha+1)a^{\frac{v+1}{2}-\alpha}}{\Gamma(\alpha)\ \Gamma(2v-\alpha+2)2^{2\alpha-v}}\ p^{2\alpha-v-1}\ e^{-p^2/4a}$, $R(\alpha)>0$, $R(v)>-1$.

3.5	$x^{\mu}\log x$	$\int_0^{\infty}\frac{y^{2\alpha-v-\mu-\frac{5}{2}}}{(1+y)^{2\alpha}}\left[\Psi\left(\frac{3v}{2}-\alpha+\frac{\mu}{2}+\frac{7}{4}\right)+\Psi\left(\alpha-\frac{\mu}{2}-\frac{v}{2}-\frac{3}{4}\right)-\log\frac{p^2y^2}{4}\right]\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]dy$ $=\frac{\Gamma(v+1)\Gamma(v-2+1)\Gamma\left(\frac{\mu+v}{2}+\frac{3}{4}\right)\Gamma\left(\alpha-\frac{\mu}{2}-\frac{v}{2}-\frac{3}{4}\right)}{2\Gamma(\alpha)\Gamma\left(\frac{3v}{2}-\alpha+\frac{\mu}{2}+\frac{7}{4}\right)\Gamma\left(\frac{v-\mu}{2}+\frac{1}{4}\right)}\left[\Psi\left(\frac{\mu+v}{2}+\frac{3}{4}\right)+\Psi\left(\frac{v-\mu}{2}+\frac{1}{4}\right)-\log\frac{p^2}{4}\right],$ $R(2\alpha-v-\mu)>\frac{3}{2},\ R(\mu+v)>-\frac{3}{2}.$
3.6	$x^{v-\frac{1}{2}}\sin(ax)$	$\int_0^{a}\frac{y^{2v+1}}{(1+y)^{2\alpha}}\,{}_2F_1\left[\begin{matrix}2v-\alpha+2\ ;\ 2v-\alpha+\frac{3}{2}\ ;\\ v+1\ ;\end{matrix}\ \frac{p^2y^2}{a^2}\right]\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]dy$ $=\frac{\sqrt{\pi}\,[\Gamma(v+1)]^2\Gamma(v-\alpha+1)\,a^{4v-2\alpha+3}}{2^{2\alpha-4v-1}\Gamma(\alpha)\Gamma(4v-2\alpha+3)\Gamma\left(\frac{1}{2}-v\right)\sin\left[\frac{\pi}{2}(4v-2\alpha+3)\right]}\ \frac{p^{2\alpha-2v-2}}{(a^2-p^2)^{v+\frac{1}{2}}},\ R(v)>-1\ ;$ $\int_a^{\infty}\frac{y^{2\alpha-2v-3}}{(1+y)^{2\alpha}}\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]\,{}_2F_1\left[\begin{matrix}2v-\alpha+2,\ v-\alpha+2\ ;\\ \frac{3}{2}\ ;\end{matrix}\ \frac{a^2}{p^2y^2}\right]=0,\ R(v)>-1.$
3.7	$x^{-\frac{3}{2}}J_v(ax)$	$\int_a^{\infty}\frac{y^{2v+1}}{(1+y)^{2\alpha}(a+py)^{4v-2\alpha+2}}\,{}_2F_1\left[\begin{matrix}2v-\alpha+1,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4apy}{(a+py)^2}\right]\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]dy$ $=\frac{[\Gamma(v+1)]^2\Gamma(v-\alpha+1)\Gamma(\alpha-v)}{2\Gamma(\alpha)v\Gamma(2v-\alpha+1)}\ p^{2\alpha-4v-2},\ R(v)>0.$
3.8	$x^{-\frac{1}{2}}\cos ax^2$	$\int_0^{\infty}\frac{y^{v+2}}{(1+y)^{2v+1}}\left[\cos\left(\frac{p^2y^2}{8a}-\frac{v\pi}{2}\right)J_{\frac{v+1}{2}}\left(\frac{p^2y^2}{8a}\right)+\sin\left(\frac{p^2y^2}{8a}-\frac{v\pi}{4}\right)J_{\frac{v-1}{2}}\left(\frac{p^2y^2}{8a}\right)\right]$ $\cdot\,{}_2F_1\left(v+\frac{1}{2},\ v+\frac{1}{2}\ ;\ 2v+1\ ;\ \frac{4y}{(1+y)^2}\right)dy=\frac{4\sqrt{\pi}\,a\Gamma(v+1)\cos\left(\frac{p^2}{8a}-\frac{v+1}{4}\pi\right)}{\Gamma\left(v+\frac{1}{2}\right)p^2}\ J_{\frac{v}{2}}\left(\frac{p^2}{8a}\right),$ $R(a)>0,\ R(v)>-1.$

3.9	$x^{-\frac{1}{2}} e^{-ax} J_v(px)$	$\int_0^\infty \frac{y^{v+1}}{(1+y)^{2\alpha}} {}_2F_1\left(\begin{matrix}\alpha, v+\frac{1}{2};\\ 2v+1;\end{matrix} \frac{4y}{(1+y)^2}\right)\left\{\sum_{m=0}^{\infty} \frac{\Gamma(4v-2\alpha+2m+3)}{m!\ \Gamma(v+m+1)}\left(-\frac{b^2}{4a^2}\right)^m {}_2F_1\left[\begin{matrix}-m, -v-m;\\ v+1;\end{matrix} \frac{p^2y^2}{b^2}\right]\right\}dy$ $=\frac{\Gamma(v+1)\Gamma(v-\alpha+1)p^{2\alpha-2v-2}}{\pi\sqrt{pb}\ \Gamma(\alpha)2^{2\alpha-2v-1}} Q_{v-\frac{1}{2}}\left(\frac{a^2+b^2+p^2}{2bp}\right),$ $R(a)>Im(b)>0,\ R(2\alpha-v)>2,\ R(v)>-2.$
3.10	$x^{-\frac{5}{2}} J_v(a/x)$	$\int_0^\infty \frac{y^v}{(1+y)^{2v}} J_{2v}(2\sqrt{apy})\ {}_2F_1\left(\begin{matrix}v, v+\frac{1}{2};\\ 2v+1;\end{matrix} \frac{4y}{(1+y)^2}\right)dy=\frac{2\Gamma(v+1)}{a\Gamma(v)p} J_{2v}(2\sqrt{ap}),\ R(v)>\frac{1}{4}.$
3.11	$x^{-\frac{5}{2}} y_v(a/x)$	$\int_0^\infty \frac{y^v}{(1+y)^{2v}}\left[K_{2v}(2\sqrt{apy})-\frac{\pi}{2}y_{2v}(2\sqrt{apy})\right] {}_2F_1\left[\begin{matrix}v, v+\frac{1}{2};\\ 2v+1;\end{matrix} \frac{4y}{(1+y)^2}\right] dy$ $=-\frac{2v}{ap}\left[K_{2v}(2\sqrt{ap})+\frac{\pi}{2}y_{2v}(2\sqrt{ap})\right],\ a>0,\ R(v)>\frac{1}{4}.$
3.12	$x^{\mu+v+\frac{1}{2}} K_\mu(ax)$	$\int_0^\infty \frac{y^{2v+1}}{(1+y)^{2\alpha}} {}_2F_1\left[\begin{matrix}v, v+\frac{1}{2};\\ 2v+1;\end{matrix} \frac{4y}{(1+y)^2}\right] {}_2F_1\left[\begin{matrix}2v+\mu-\alpha+2, 2v-\alpha+2;\\ v+1;\end{matrix} -\frac{p^2y^2}{a^2}\right] dy$ $=\frac{[\Gamma(v+1)]^2\Gamma(v-\alpha+1)\Gamma(\mu+v+1)\ a^{4v-2\alpha+2\mu+4}\ p^{2\alpha-2v-2}}{2\Gamma(2v+\mu-\alpha+2)\Gamma(2v-\alpha+2)\Gamma(\alpha)\ (p^2+a^2)^{\mu+v+1}},\ R(a)>0,\ R(v)>-1,$ $R(\mu+v+1)>0.$
3.13	$x^{2\mu+v+\frac{1}{2}}$ $\exp\left(-\frac{a2x2}{4}\right)K_\mu\left(\frac{a2x2}{4}\right)$	$\int_0^\infty \frac{y^{2\alpha-2v-2\mu-3}}{(1+y)^{2\alpha}} {}_2F_1\left[\begin{matrix}\alpha, v+\frac{1}{2};\\ 2v+1;\end{matrix} \frac{4y}{(1+y)^2}\right] G_{23}^{12}\left[\frac{y^2p^2}{2a^2} \,\middle\vert\, \begin{matrix}1-\mu, 1+\mu\\ 2v+\mu-\alpha+2, \frac{1}{2}, v+\mu-\alpha+2\end{matrix}\right] dy$ $=\frac{\Gamma(v+1)\Gamma(v-\alpha+1)\Gamma(1+2\mu+v)}{\Gamma(\alpha)\Gamma\left(\mu+v+\frac{3}{2}\right)2^{\mu+v+2}\ a^{2(\mu+v+1)}}\ {}_1F_1\left[\begin{matrix}1+2\mu+v;\\ \mu+v+\frac{3}{2};\end{matrix} -\frac{p^2}{2a^2}\right],\ R(v)>-1,\ R(2\alpha-v)>\frac{3}{2},$ $\lvert \arg a\rvert<\frac{\pi}{4},\ R(2v-\alpha+2)>0,\ R(2v+2\mu-\alpha+2)>0.$

3.14	$H_{v-1/2}(ax)$	$\int_0^a \frac{y^{2\alpha-v-\frac{5}{2}}}{(1+y)^{2\alpha}}\ {}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right] G_{33}^{21}\left(\frac{p^2y^2}{a^2}\left\vert\begin{matrix}\frac{3}{4}-\frac{v}{2},\ \frac{5}{4}-\frac{v}{2},\ \frac{v}{2}+\frac{3}{4}\\ \frac{3v}{2}-\alpha+\frac{7}{4},\ \frac{3}{4}-\frac{v}{2},\ \frac{v}{2}-\alpha+\frac{7}{4}\end{matrix}\right.\right) dy$ $=\frac{a^{v-\frac{1}{2}}\Gamma(v+1)\Gamma(v-\alpha+1)p^{\frac{3}{2}-v}}{2\sqrt{\pi}\Gamma(\alpha)\ (a^2-p^2)^{\frac{1}{2}}}$, $a>0$, $R(2v-\alpha+2)>0$, $R(2\alpha-2v)>5/2$, $R(2\alpha-3v)>\frac{3}{2}$, $-1<R(v)<\frac{3}{2}$ ($R(v)<\frac{3}{2}$ *can be waved*)
3.15	$\sqrt{x}\,[H_{-v}(ax)$ $-y_{-v}(ax)]$	$\int_0^\infty \frac{y^{2\alpha-v-3}}{(1+y)^{2\alpha}}\ {}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right] G_{33}^{23}\left(\frac{p^2y^2}{a^2}\left\vert\begin{matrix}\frac{1+v}{2},\ 1+\frac{v}{2},\ 1-\frac{v}{2}\\ \frac{3v}{2}-\alpha+2,\ \frac{1+v}{2},\ \frac{v}{2}-\alpha+2\end{matrix}\right.\right) dy$ $=\frac{\pi\Gamma(v+1)\Gamma(v-\alpha+1)}{2a^v\Gamma(\alpha)}\ \frac{p^{v+1}}{p+a}$, $R(v-\alpha)>-2$, $R(2v-\alpha)>-2$, $R\left(v-\alpha+\frac{3}{2}\right)<0$, $-1<R(v)<1$.
3.16	$\left[H_{-v-1}\left(\frac{a}{x}\right)-y_{-v-1}\left(\frac{a}{x}\right)\right]\cdot x^{-\frac{3}{2}}$, $\lvert R(v)\rvert<\frac{1}{2}$.	$\int_0^\infty \frac{y^{2\alpha-v-1}}{(1+y)^{2\alpha}}\ {}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right] G_{15}^{41}\left(\frac{a^2p^2y^2}{16}\left\vert\begin{matrix}-\frac{v}{2}\\ \frac{3v}{2}-\alpha+1,\ \frac{v+1}{2},\ -\frac{v}{2},\ -\frac{v+1}{2},\ \frac{v}{2}-\alpha+1\end{matrix}\right.\right) dy$ $=\frac{4\pi\,\Gamma(v+1)\Gamma(v-\alpha+1)}{\sqrt{a}\,\Gamma(\alpha)\,\sqrt{p}}\ K_{-2v-1}(2\sqrt{ap})$, $R(\alpha)>0$, $R(2\alpha-3v)>\frac{3}{2}$, $R(4v-2\alpha)>-4$, $0>R(v)>-1$, $R\left(2\alpha-v-\frac{1}{2}\right)>0$.
3.17	$x^{\mu-v+\frac{1}{2}}[I_\mu(ax)-$ $L_\mu(ax)]$	$\int_0^\infty \frac{y^{2\alpha-\mu-3}}{(1+y)^{2\alpha}}\ {}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right] G_{33}^{22}\left(\frac{p^2y^2}{a^2}\left\vert\begin{matrix}1-\frac{\mu}{2},\ \frac{1-\mu}{2},\ 1+\frac{\mu}{2}\\ v-\alpha+\frac{\mu}{2}+2,\ \frac{1-\mu}{2},\ \frac{\mu}{2}-\alpha+2\end{matrix}\right.\right) dy$ $=\frac{\sqrt{\pi}\ a^{\mu-1}\Gamma(v+1)\Gamma(v-\alpha+1)}{2\Gamma\left(v-\mu+\frac{1}{2}\right)\Gamma(\alpha)\,p^{\mu-1}}\ {}_2F_1\left(1,\ \frac{1}{2}\ ;\ v-\mu+\frac{1}{2}\ ;\ -\frac{p^2}{a^2}\right)$, $R(a)>0$; $R(v)>-1$, $R(v-\alpha+\mu)>-2$, $R(2\alpha-v-2\mu)>\frac{3}{2}$, $R(2\alpha-v)>\frac{3}{2}$.

3.18	$x^{v-\frac{1}{2}}e^{-\frac{a^2x^2}{4}}D_{2\mu}(ax)$	$\int_0^\infty \frac{y^{2v+1}}{(1+y)^{2\alpha}}\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]\,{}_2F_2\left[\begin{matrix}2v-\alpha+\frac{3}{2},\ 2v-\alpha+2\ ;\\ v+1,\ \frac{3v}{2}-\mu-\alpha+2\ ;\end{matrix}\ -\frac{p^2y^2}{4}\right]dy$ $=\frac{2^{5v-3\alpha+2}\Gamma\left(v+\frac{1}{2}\right)[\Gamma(v+1)]^2\Gamma(v-\alpha+1)\Gamma\left(\frac{3v}{2}-\alpha-\mu+2\right)a^{2v-2\alpha+2}}{\sqrt{\pi}\,\Gamma(4v-2\alpha+3)\Gamma(v-\mu+1)\Gamma(\alpha)p^{-2\alpha+2v+2}}$ $\cdot\,{}_1F_1\left(v+\frac{1}{2}\ ;v-\mu+1\ ;-\frac{p^2}{2a^2}\right),\ R(v)>-\frac{1}{2}.$
3.19	$x^{2\mu-v-\frac{1}{2}}\exp\left(-\frac{x^2}{4}\right)$ $M_{K,\mu}\left(\frac{x^2}{2}\right)$	$\int_0^\infty \frac{y^{K+\mu+v-\frac{1}{2}}}{(1+y)^{4\mu+2}}e^{-p^2\frac{y^2}{4}}W_{\frac{K}{2}-\frac{3\mu}{2}+\frac{v}{2}+\frac{1}{4},\ -\frac{K}{2}+\frac{\mu}{2}-\frac{v}{2}-\frac{1}{4}}\left(\frac{p^2y^2}{2}\right)\,{}_2F_1\left[\begin{matrix}2\mu+1,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]dy$ $=\frac{2^{v-\mu-1}\Gamma(v+1)\Gamma(v-2\mu)\Gamma\left(\mu+K+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+K-\mu+v\right)\Gamma(2\mu+1)p^{2v-2\mu}}e^{-\frac{p^2}{4}}M_{\frac{1}{4}+\frac{K}{2}+\frac{3\mu}{2}-\frac{v}{2},\ -\frac{1}{4}+\frac{K}{2}-\frac{\mu}{2}+\frac{v}{2}}(p^2/2),$ $R(\mu)>-\frac{1}{2},\ R(K+v)>-1.$
3.20	$x^{v-\frac{1}{2}-2\mu}\exp\left(-\frac{x^2}{4}\right)$ $W_{K,\mu}\left(\frac{x^2}{2}\right)$	$\int_0^\infty \frac{y^{2v+1}}{0(1+y)^{2\alpha}}\,{}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\ ;\\ 2v+1\ ;\end{matrix}\ \frac{4y}{(1+y)^2}\right]\,{}_2F_2\left[\begin{matrix}2v-\alpha+2,\ 2v-2\mu-\alpha+2\ ;\\ \frac{5}{2}-K+2v-\mu-\alpha,\ v+1\ ;\end{matrix}\ -\frac{p^2y^2}{2}\right]dy$ $=\frac{\Gamma(1+v-2\mu)[\Gamma(v+1)]^2\Gamma\left(\frac{5}{2}-K-\mu-\alpha\right)\Gamma(v-\alpha+1)p^{K-3v+\mu+2\alpha-\frac{7}{2}}}{\Gamma(\alpha)\Gamma(2v-\alpha+2)\Gamma(2v-2\mu-\alpha+2)\Gamma(1+2\beta)2^{\frac{K}{2}-\frac{3v}{2}+\frac{\mu}{2}+\alpha-\frac{3}{4}}}$ $\cdot\,e^{-\frac{p^2}{4}}M_{\alpha}1,\ {}_{\beta}\left(\frac{p^2}{2}\right),\ R(v)>-1,\ R(v-2\mu)>-1$; *where* $2\alpha'=\frac{1}{2}+K+v-\frac{3}{2},\ 2\beta=\frac{1}{2}-K+v-\mu.$

		In particular, if $K=\mu+\frac{1}{2}$, *we get* $$\int_0^\infty \frac{y^{2v+1}}{(1+y)^{2\alpha}}\, {}_2F_1\left[\begin{matrix}\alpha,\ v+\frac{1}{2}\,;\\ 2v+1\,;\end{matrix}\ \frac{4y}{(1+y)^2}\right] {}_2F_2\left[\begin{matrix}2v-\alpha+2,\ 2v-2\mu-\alpha+2\,;\\ 2+2v-2\mu-\alpha,\ v+1\,;\end{matrix}\ -\frac{p^2y^2}{2}\right] dy$$ $$=\frac{\Gamma(1+v-2\mu)\,[\Gamma(v+1)]^2\,\Gamma(2+2v-2\mu-\alpha)\,\Gamma(v-\alpha+1)\,p^{2\alpha-2v-2}}{\Gamma(\alpha)\,\Gamma(2v-\alpha+2)\,\Gamma(2v-2\mu-\alpha+2)\,\Gamma(1-2\mu+v)\,2^{\alpha-v}}\,e^{-p^2/2}$$ $$R(\alpha)>0,\ R(v)>-1,\ R(v-2\mu)>-1.$$

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