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THE STRUCTURE OF MULTI-BLOCK CIRCULANTS

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1. Introduction

In this paper we extend a result of Trapp [4] for block circulant matrices to multi-block circulant matrices. A o-block circulant is simply a circulant matrix; that is, a matrix of the form

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \cdots & & & \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

A 1-block circulant, or simply block circulant, is a partitioned matrix of the form

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & \cdots & A_{n-2} \\ \cdots & & & \\ A_1 & A_2 & \cdots & A_0 \end{pmatrix}$$

where all A_i, for $i=0, 1 \cdots, n-1$, are circulants of the same size. In general, an m-block circulant is a partitioned matrix

$$\begin{pmatrix} M_0 & M_1 & \cdots & M_{n-1} \\ M_{n-1} & M_0 & \cdots & M_{n-2} \\ \cdots & & & \\ M_1 & M_2 & \cdots & M_0 \end{pmatrix}$$

where each M_i for $i=0, 1, \cdots, n-1$, is an $(m-1)$-block circulant and all M_i have the same size.

Circulants and multi-block circulants have many applications in physics, Fourier analysis, geometry, probability and statistics (see [2] and [3]).

Our main interst here is to describe the inverse of a multi block circulant.

2. Notation

We denote the $n_0 \times n_0$ circulant matrix

$$A(0)=\begin{pmatrix} a_0 & a_1 & \cdots & a_{n_0-1} \\ a_{n_0-1} & a_0 & \cdots & a_{n_0-2} \\ \cdots & & & \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

as circ $[a_0, a_1, \cdots, a_{n_0-1}]$.

Likewise we describe the $n_1 \times n_1$ 1-block circulant

$$A(1)=\begin{pmatrix} A_0(0) & A_1(0) & \cdots & A_{n_1-1}(0) \\ A_{n_1-1}(0) & A_0(0) & \cdots & A_{n_1-2}(0) \\ \cdots & & & \\ A_1(0) & A_2(0) & \cdots & A_0(0) \end{pmatrix}$$

as circ $[A_0(0), A_1(0), \cdots, A_{n_1-1}(0)]$, where each $A_i(0)$, $i=0, 1, \cdots, n_1-1$, is an $n_0 \times n_0$ circulant matrix. Continuing in this fashion we describe an $n_k \times n_k$ k-block circulant matrix as

$$A(k)=\text{circ } [A_0(k-1), A_1(k-1), \cdots, A_{n_k-1}(k-1)],$$

where each $A_i(k-1)$, $i=0, 1, \cdots, n_k-1$ is a $(k-1)$-block circulant matrix.

For the statement of the main theorem we will need the following matrices:

1) For $k \geq 0$, $\Omega(k)$ is the $n_k \times n_k$ matrix $\Omega(k)=(p_{n_k}^{\ ij})$ $i, j=0, 1, \cdots, n_k-1$ where

$$p_{n_k}=e^{\frac{2\Pi i}{n_k}}$$

These elements are roots of unity.

2) For $k \geq \phi$ define $\Gamma(k)$ recursively by $\Gamma(k)=\Omega(k)\otimes\Gamma(k-1)/\text{SQRT}(n_k)$ where $\otimes$ denotes the Kronecker product (see Davis [2], pp. 22~24) and where $P(-1)=(1)$.

The diagonalization of a multi-block circulant

THEOREM 1. *Let $A(m)$ be an m-block circulant matrix. Then we have $\Gamma^{-1}(m) A(m)\Gamma(m)=D(m)$ where $D(m)$ is a block diagonal matrix whose diagonal blocks $D_0(m-1)$, $D_1(m-1)$, $\cdots$, $D_{n_m-1}(m-1)$ are themselves diagonal matrices. The diagonal elements of $D_i(m-1)$, for $i=0, 1, \cdots, n_m-1$, are the eigenvalues of the matrix*

$$\sum_{k=0}^{n_m-1} A_k(m-1) r_k^i$$

where $r_0, r_1, \cdots, r_{n_m-1}$ are the n_m-th roots of unity ($r_i=p_{n_m}^i$, $i=0, 1, \cdots, n_m-1$).

The proof is by induction on m. The case $m=1$ has been proven by Trapp [4] and Chao [1].

Assume the theorem is true for all $(m-1)$-block circulant matrices. Consider an m-block circulant matrix $A(m)=\text{circ}\ [A_0(k), A_1(k), \cdots, A_{n-1}(k)]$ where $k=m-1$ and $A_i(k)$ is a k-block circulant for each $i=0, 1, \cdots, n_m-1$.

Let $\Omega=\Omega(m)$ and $\Gamma=\Gamma(k)$ where the matrices $\Gamma(k)$ and $\Omega(m)$ have been described previously. The matrix $A(m)$ can be written as

$$A(m)=I\otimes A_0(k)+T\otimes A_1(k)+T^2\otimes A_2(k)+\cdots+T^{n_m-1}\otimes A_{n_m-1}(k)$$

where T is the $n_m\times n_m$ permutation matrix that corresponds to the cyclic permutation

$$\begin{bmatrix} 0 & 1 & \cdots & i & \cdots & n_m-1 \\ 1 & 2 & \cdots & i+1 & \cdots & 0 \end{bmatrix}$$

We have $\Gamma(m)^{-1}A(m)\Gamma(m)=(\Omega\otimes\Gamma)^{-1}A(m)(\Omega\otimes\Gamma)$

$$\begin{aligned}
&=(\Omega^{-1}\otimes\Gamma^{-1})\ (I\otimes A_0(k)+T\otimes A_1(k)+\cdots+T^{n_m-1}\otimes A_{n_m-1}(k))\ (\Omega\otimes\Gamma)\\
&=(\Omega^{-1}I\Omega)\otimes(\Gamma^{-1}A_0(k)\Gamma)+(\Omega^{-1}T\Omega)\otimes(\Gamma^{-1}A_1(k)\Gamma)\\
&\quad+\cdots+(\Omega^{-1}T^{n_m-1}\Omega)\otimes(\Gamma^{-1}A_{n_m-1}(k)\Gamma)
\end{aligned}$$

Since $A_i(k)$ is a k-block circulant, the induction hypotheses gives, $\Gamma^{-1}A_i(k)\Gamma=E_i$, where E_i is a diagonal matrix. Since $A_i(k)$ is similar to a diagonal matrix, E_i, the eigenvalues of $A_i(k)$ are the diagonal entries of E_i.

Each T^i, $i=0, 1, \cdots, n_m-1$, is an $n_m\times n_m$ circulant matrix. By Chao [1]

$$\begin{aligned}
\Omega^{-1}T^i\Omega&=\text{diag}\ \{P^0_{n_m}, P^i_{n_m}, P^{2i}_{n_m}, \cdots P^{(n_m-1)i}_{n_m}\}\\
&=\text{diag}\ \{r_i^0, r_i, r_i^2, \cdots r_i^{n_m-1}\}=R_i
\end{aligned}$$

Now

$$\begin{aligned}
&(\Omega\otimes\Gamma)^{-1}A(m)(\Omega\otimes\Gamma)\\
&=I\otimes(\Gamma^{-1}A_0(k)\Gamma)+(\Omega^{-1}T\Omega)\otimes(\Gamma^{-1}A_1(k)\Gamma)\\
&\quad+\cdots+(\Omega^{-1}T^{n_m-1}\Omega)\otimes(\Gamma^{-1}A_{n_m}(k)\Gamma)\\
&=I\otimes E_0+R_1\otimes E_1+R_2\otimes E_2+\cdots+R_{n_m-1}\otimes E_{n_m-1}\\
&=\text{diag}\ \{\sum_{i=0}^{n_m-1}E_i, \sum_{i=0}^{n_m-1}r_iE_i, \sum_{i=0}^{n_m-1}r_i^2E_i, \cdots, \sum_{i=0}^{n_m-1}r_i^{n_m-1}E_i\}\\
&=\text{diag}\ \{D_0(k), D_1(k), \cdots, D_{n_m-1}(k)\}
\end{aligned}$$

where, for each $i=0, 1, \cdots, n_m-1$, $D_i(k)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\sum_{j=0}^{n_m-1} r_j^i A_j(k)$. Hence

$$\Gamma^{-1}(m)A(m)\Gamma(m)=D(m)=\text{diag}\ \{D_0(m-1),\ D_1(m-1),\ \cdots,\ D_{n_m-1}(m-1)\}$$

where $D_i(m-1)$, for each $i=0, 1, \cdots, n_m-1$, is a diagonal matrix whose diagonal entries are the eigenvalues of

$$\sum_{k=0}^{n_m-1} r_k A_k(m-1)^i$$

Our next theorem describes how to "recover" an m-block circulant from a diagonal matrix.

THEOREM 2. *Given a square diagonal matrix whose size is* $n_m \cdot n_{m-1} \cdots n_1 \cdot n_0$, *then* $\Gamma(m)D\Gamma^{-1}(m)$ *is an m-block circulant matrix.*

PROOF. The proof proceeds by induction on m. The case $m=1$ has been proven by Trapp [4] and Chao [1].

Let $D=\text{diag}\ \{D_0, D_1, \cdots, D_{n_m-1}\}$ where D_i, for each $i=0, 1, \cdots, n_m-1$ is a square diagonal matrix whose size is $n_0 \cdot n_1 \cdots n_{m-1}$. Again let $\Omega=\Omega(m)$ and $\Gamma=\Gamma(m-1)$, $\Gamma(m)D\Gamma^{-1}(m)$ can be written as

$$\begin{aligned}\Gamma(m)D\Gamma^{-1}(m) &= (\Omega\otimes\Gamma)D(\Omega\otimes\Gamma)^{-1}\\
&= (\Omega\otimes\Gamma)\ (\text{diag}\ (1, 0, \cdots, 0)\otimes D_0+\text{diag}\ (0, 1, 0, \cdots, 0)\otimes D_1\\
&\quad +\cdots+\text{diag}\ (0, 0, \cdots 0, 1)\otimes D_{n_m-1})\ (\Omega^{-1}\otimes\Gamma^{-1})\\
&= (\Omega\ \text{diag}\ (1, 0, \cdots, 0)\Omega^{-1})\otimes(\Gamma D_0\Gamma^{-1})\\
&\quad +(\Omega\ \text{diag}\ (0, 1, 0, \cdots, 0)\Omega^{-1})\otimes(\Gamma D_1\Gamma^{-1})\\
&\quad +\cdots+(\Omega\ \text{diag}\ (0, 0, \cdots, 0, 1)\Omega^{-1})\otimes(\Gamma D_{n_m-1}\Gamma^{-1})\end{aligned}$$

For each $i=0, 1, \cdots, n_m-1$, the induction hypothesis shows that $\Gamma D_i\Gamma^{-1}$ is an $(m-1)$-block circulant. We also have the (i, j) entry of Ω diag$(0, \cdots, 0, 1, 0, \cdots, 0)$ $\Omega^{-1}=r^{ik}(\bar{r})^{jk}$, here the one in the diagonal matrix is in the k-th position. The (i, j) block of $\Gamma(m)D\Gamma^{-1}(m)$ is

$$\sum_{k=0}^{n_m-1} r^{ik}(\bar{r})^{jk}\ A_k(m-1)$$

and the $(i+1,\ j+1)$ block of $\Gamma(m)D\Gamma^{-1}(m)$ is

$$\sum_{k=0}^{n_m-1} r^{(i+1)k}(\bar{r})^{(j+1)k}\ A_k(\mathrm{m}-1)$$
$$=\sum_{k=0}^{n_m-1} r^{ik}(\bar{r})^{jk}(r\bar{r})^k\ A_k(m-1)$$
$$=\sum_{k=0}^{n_m-1} r^{ik}(\bar{r})^{jk}\ A_k(m-1)$$

Hence the $(i,\ j)$ block equals the $(i+1,\ j+1)$ block. This equality also holds if i or j is n_m-1 and we take $i+1$ or $j+1$ modulo n_m.

Hence $\Gamma(m)D\Gamma^{-1}(m)$ is an m-block circulant matrix.

Our final result is a direct consequence of the above two theorems.

THEOREM 3. *If $A(m)$ is non-singular and an m-block circulant matrix, then $A^{-1}(m)$ is also an m-block circulant matrix.*

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