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SOME PROPERTIES OF BITOPOLOGICAL HYPERSPACES

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1. Introduction

In this paper we continue the study of topological and bitopological properties considered in [1], [3], [4], [5], [6] and [10]. We use the same terms and notations as in [3] and [5] but without the assumption that at least one of the spaces (X, τ^i), $i=1,2$, is a T_1 space.

In a bitopological space (X, τ^1, τ^2) a set is said to be *quasiopen* if it is a union of τ^1 open and τ^2 open sets. A set is quasiclosed if its complement is quasiopen. A set is *semiopen* or *τ open* (*semiclosed* or *τ closed*) if it is open (closed) in the topological space (X, τ), where $\tau=\tau^1\vee\tau^2$, is the least upper bound topology of τ^1 and τ^2. A space (X, τ^1, τ^2) is *quasicompact* if the topological space (X, τ) is compact. The closure of a subset A in a topological space (X, τ^i) is denoted by $\tau^i\text{cl } A$. The *quasiclosure* of a subset A in a bitopological space (X, τ^1, τ^2) is $\text{qcl } A=\tau^1\text{cl } A\cap\tau^2\text{cl } A$. Notions of pairwise T_k and weak pairwise T_k spaces for $k=0,1,2$ and 3 coincide with those in [12].

If (X, τ^1, τ^2) is an arbitrary bitopogical space, then

$\mathscr{A}(X)=\{A\subset X \mid A\neq\phi\}$,

$\mathscr{S}(X)=\{A\in\mathscr{A}(X) \mid A \text{ is } \tau \text{ closed}\}$,

$\mathscr{Q}(X)=\{A\in\mathscr{A}(X) \mid A \text{ is quasiclosed}\}$,

$\mathscr{K}(X)=\{A\in\mathscr{A}(X) \mid A \text{ is } \tau^1 \text{ or } \tau^2 \text{ closed}\}$,

$\mathscr{B}(X)=\{A\in\mathscr{A}(X) \mid A \text{ is } \tau^1 \text{ and } \tau^2 \text{ closed}\}$,

$\mathscr{C}^i(X)=\{A\in\mathscr{A}(X) \mid A \text{ is } \tau^i \text{ compact}\}$, for $i=1,2$,

$\mathscr{C}^*(X)=\mathscr{C}^1(X)\cap\mathscr{C}^2(X)$,

$\mathscr{F}_n(X)=\{A\in\mathscr{A}(X) \mid A \text{ has at most } n \text{ elements}\}$.

If $\mathscr{F}$ is one of the collections $\mathscr{B}(X)$, $\mathscr{K}(X)$, $\mathscr{Q}(X)$, $\mathscr{S}(X)$ or $\mathscr{A}(X)$ then

$\mathscr{C}\mathscr{F}=\{A\in\mathscr{F} \mid A \text{ is quasicompact}\}$,

$\mathscr{C}^*\mathscr{F}=\{A\in\mathscr{F} \mid A \text{ is } \tau^1 \text{ and } \tau^2 \text{ compact}\}$.

If $\mathscr{F}$ is any collection of non-empty subsets of X, for any subset $V\subset X$ we consider the following subcollections:

$$<V>=\{A\in\mathscr{F}\mid A\subset V\} \text{ and } >V<=\{A\in\mathscr{F}\mid A\cap V\neq\phi\}.$$

In particular, the families

$$\overline{\mathscr{P}}^i=\{<V>\mid V\in\tau^i\},\ \underline{\mathscr{P}}^i=\{>V<\mid V\in\tau^i\} \text{ and } \mathscr{P}=\overline{\mathscr{P}}^i\cup\underline{\mathscr{P}}^i$$

are subbases for topologies on $\mathscr{F}$ induced by τ^i in X; these are the upper semi-finite, the lower semi-finite and the finite (Vietoris) topology, respectively denoted by $\overline{\mathscr{T}}^i$, $\underline{\mathscr{T}}^i$ and $\mathscr{T}^i$, The family of subcollections

$$<V_0\,;V_1,\cdots,V_n>=\{A\in\mathscr{F}\mid A\subset V_0=\bigcup_{k=1}^{n} V_k \text{ and } A\cap V_k\neq\phi \text{ for each } k=1,\cdots,n\}$$

where $V_k\in\tau^i$ and $n\in\boldsymbol{N}$ is a basis for the topology $\mathscr{T}^i$.

In [6], in a bitopological space $(X,\ \tau^1,\ \tau^2)$ the equivalence relation $\sim$ was defined by

$$x\sim y \Leftrightarrow \mathrm{qcl}\{x\}=\mathrm{qcl}\{y\}.$$

As in [6], let $[x]$ denote the equivalence class of the element x of X, $\tilde{X}$ be the quotient set under the relation $\sim$, $\tilde{\tau}^1$, $\tilde{\tau}^2$ and $\tilde{\tau}$ the quotient topologies in $\tilde{X}$ of τ^1, τ^2 and τ respectively, and $p:X\to\tilde{X}$ the quotient mapping. It was shown [6, Theorem 1] that many of the bitopological properties are shared by X and $\tilde{X}$.

We make extensive use of the following results from [6] which are restated here for convenience.

PROPOSITION A ([6, Corollary 1]). *For each semiopen (semiclosed) subset A of X, $p^{-1}(p(A))=A$ holds.*

PROPOSITION B ([6, Proposition 4]). *The topology $\tilde{\tau}$ is the least upper bound topology of $\tilde{\tau}^1$ and $\tilde{\tau}^2$, i.e. $\tilde{\tau}=\tilde{\tau}^1\vee\tilde{\tau}^2$.*

PROPOSITION C ([6, Corollary 2]). *The quotient mapping p is pairwise open and pairwise closed.*

PROPOSITION D ([6, Corollary 3]). *For each $x\in X$, the set $[x]$ is quasicompact in X, hence τ^1 and τ^2 compact.*

PROPOSITION E ([6, Proposition 5]). *For each $A\subset X$ we have*

$$\tilde{\tau}^i\mathrm{cl}\ p(A)=p(\tau^i\mathrm{cl}\ A) \text{ for } i=1,\ 2, \text{ and } \mathrm{qcl}(p(A))=p(\mathrm{qcl}\ A).$$

PROPOSITION F ([6, Proposition 9]). *If a bitopological space $(X,\ \tau^1,\ \tau^2)$ is pairwise R_0, the topological space $(X,\ \tau)$ is an R_0 space.*

PROPOSITION G ([6, Proposition 11]). *Every weak pairwise T_1 pairwise R_1*

bitopological space is weak pairwise T_2.

2. Pairwise homeomorphism of bitopological hyperspaces

In the sequel the same notation is used for a topology on a set and for the relative topology on its subset.

THEOREM 3. *Let* (X, τ^1, τ^2) *be a bitopological space,* $(\tilde{X}, \tilde{\tau}^1, \tilde{\tau}^2)$ *be its quotient space and* $\mathscr{F}$ *one of collections* $\mathscr{S}(X)$, $\mathscr{C}(X)$, $\mathscr{K}(X)$ *or* $\mathscr{B}(X)$. *If* θ^i *is one of topologies* $\mathscr{T}^i$, $\overline{\mathscr{T}}^i$ *or* $\underline{\mathscr{T}}^i$ *for* $i=1, 2$, *then the corresponding hyperspaces* $(\mathscr{F}, \theta^1, \theta^2)$ *and* $(\tilde{\mathscr{F}}, \tilde{\theta}^1, \tilde{\theta}^2)$ *are pairwise homeomorphic.*

PROOF. (i) We prove first that the hyperspaces $(\mathscr{S}(X), \theta^1, \theta^2)$ and $(\mathscr{S}(\tilde{X}), \tilde{\theta}^1, \tilde{\theta}^2)$ are pairwise homeomorphic. Let $q : \mathscr{S}(X) \to \mathscr{S}(\tilde{X})$ be defined by the quotient mapping p, i.e. for each $A \in \mathscr{S}(X)$, $q(A) = p(A)$. The mapping q is the restriction of the bijection q^* from Proposition 6 of [6]. The bijection q is pairwise continuous since for each $\tilde{\tau}^i$ open subset $\tilde{V}$ of $\tilde{X}$ and the corresponding subbasic elements of the topology $\tilde{\overline{\mathscr{T}}}$, $i=1, 2$, we have

$$q^{-1}(<\tilde{V}>) = \{A \in \mathscr{S}(X) \mid p^{-1}(p(A)) \subset p^{-1}(\tilde{V})\} = <p^{-1}(\tilde{V})> \in \overline{\mathscr{T}}^i$$

and

$$q^{-1}(>\tilde{V}<) = \{A \in \mathscr{S}(X) \mid p^{-1}(p(A)) \cap p^{-1}(\tilde{V}) \neq \phi\} = >p^{-1}(\tilde{V})< \in \underline{\mathscr{T}}^i.$$

Similarly, q is a pairwise open mapping since for each τ^i open subset V of X and the corresponding subbasic elements of the topology τ^i, $i=1,2$,

$$q(<V>) = <p(V)> \in \tilde{\overline{\mathscr{T}}}^i \text{ and } q(>V<) = >p(V)< \in \tilde{\underline{\mathscr{T}}}^i$$

holds.

(ii) The pairwise homeomorphism $q : \mathscr{S}(X) \to \mathscr{S}(\tilde{X})$ induces a pairwise homeomorphism of subspaces $\mathscr{C}(X)$ and $\mathscr{C}(\tilde{X})$. Furthermore, since p is a pairwise continuous and pairwise closed mapping, $q(\mathscr{K}(X)) = \mathscr{K}(\tilde{X})$ and $q(\mathscr{B}(X)) = \mathscr{B}(\tilde{X})$ holds. Hence q induces pairwise homeomorphisms of the corresponding subspaces.

COROLLARY 1. *Under the notations given in the previous theorem, the subspaces* $(\mathscr{C}\mathscr{F}, \theta^1, \theta^2)$ *and* $(\mathscr{C}\tilde{\mathscr{F}}, \tilde{\theta}^1, \tilde{\theta}^2)$, $((\mathscr{C}^*\mathscr{F}, \theta^1, \theta^2)$ *and* $(\mathscr{C}^*\tilde{\mathscr{F}}, \tilde{\theta}^1, \tilde{\theta}^2))$ *are pairwise homeomorphic.*

3. Hyperspaces of pairwise R_0 and pairwise R_1 spaces

Some properties of hyperspaces of topological R_0 and R_1 spaces have been investigated in [1] and [4]. We consider here analogous questions in the case of pairwise R_0 and pairwise R_1 bitopological spaces.

Propositions F, E and D imply the following result.

COROLLARY 2. *If* (X, τ^1, τ^2) *is a pairwise* R_0 *space, then* $\mathcal{J}_1(\tilde{X})\subset\mathcal{C}\mathcal{Q}(\tilde{X})\subset\mathcal{C}^*\mathcal{Q}(\tilde{X})\subset\mathcal{Q}(\tilde{X})$, *i.e.* $\tilde{X}\subset\mathcal{C}\mathcal{Q}(X)\subset\mathcal{C}^*\mathcal{Q}(X)\subset\mathcal{Q}(X)$.

The collection $\mathcal{Q}(X)$ cannot be replaced by $\mathcal{K}(X)$ even in the case X is pairwise R_1 or pairwise regular, as Example 1 shows. Example 1 of [7] shows that the converse statement of the previous corollary does not hold in general.

EXAMPLE 1. Let $X=\boldsymbol{R}$, the set of real numbers, τ^1 be the left hand topology and τ^2 the right hand topology. $\mathcal{K}(X)=\{(-\infty, a]\mid a\in\boldsymbol{R}\}\cup\{[a, +\infty)\mid a\in\boldsymbol{R}\}\cup\{\boldsymbol{R}\}$, while for each $x\in\boldsymbol{R}$, $[x]=\{x\}$.

PROPOSITION 1. *Let* (X, τ^1, τ^2) *be pairwise* R_0. *For each* $F\in\mathcal{Q}(X)$, $\mathcal{T}^1\mathrm{cl}\{F\}\cap\mathcal{T}^2\mathrm{cl}\{F\}=\{F\}$, *i.e. the quasiclosure of* $\{F\}$ *in* $(\mathcal{Q}(X), \mathcal{T}^1, \mathcal{T}^2)$ *is* $\{F\}$.

PROOF. Let F, $H\in\mathcal{Q}(X)$ and $F\neq H$. Then $F=F_1\cap F_2$ and $H=H_1\cap H_2$, where F_k and H_k are τ^k closed for $k=1, 2$. There is an $x\in(F-H)\cup(H-F)$. (*i*) Let $x\in F-H$. There exists an $i\in\{1,2\}$ such that $x\notin H_i$. Then $x\in H_i^c$ and $\tau^j\mathrm{cl}\{x\}\subset H_i^c$, for $j\in\{1,2\}-\{i\}$. Let $V=(\tau^j\mathrm{cl}\{x\})^c$. Then $V\in\tau^j$, $H\subset H_i\subset V$ and $F\not\subset V$. Let $\mathcal{V}=\langle V\rangle\in\bar{\mathcal{T}}^j\subset\mathcal{T}^j$. Since $H\in\mathcal{V}$ and $F\notin\mathcal{V}$, $H\notin\mathcal{T}^j\mathrm{cl}\{F\}$, so $H\notin\mathrm{qcl}\{F\}$.

(*ii*) *If* $x\in H-F$, there is an $i\in\{1, 2\}$ such that $x\in H$ and $x\notin F_i$. Let $U=F_i^c$, then $U\in\tau^i$ and $\mathcal{U}=\rangle U\langle\in\underline{\mathcal{T}}^i\subset\mathcal{T}^i$. It follows that $H\in\mathcal{U}$, $F\notin\mathcal{U}$, so $H\notin\mathcal{T}^i\mathrm{cl}\{F\}$. Hence, $H\notin\mathrm{qcl}\{F\}$.

COROLLARY 3. *If* (X, τ_1, τ_2) *is pairwise* R_0, *then for each* $F\in\mathcal{Q}(X)$ *the set* $\{F\}$ *is semiclosed in* $(\mathcal{Q}(X), \mathcal{T}^1, \mathcal{T}^2)$, *i.e. it is* $\mathcal{T}$ *closed in* $\mathcal{Q}(X)$, *where* $\mathcal{T}=\mathcal{T}^1\vee\mathcal{T}^2$ *is the least upper bound topology of* $\mathcal{T}^1$ *and* $\mathcal{T}^2$.

COROLLARY 4. *If* (X, τ^1, τ^2) *is pairwise* R_0 *and* $\mathcal{F}$ *is one of the collections* $\mathcal{C}\mathcal{Q}(X)$ *or* $\mathcal{C}^*\mathcal{Q}(X)$, *then for each* $E\in\mathcal{F}$ *the set* $\{F\}$ *is quasiclosed (semiclosed) in* $(\mathcal{F}, \mathcal{T}^1, \mathcal{T}^2)$.

In Proposition 1 and Corollaries 3 and 4 the collection $\mathcal{Q}(X)$ cannot be replaced by $\mathcal{S}(X)$ in general. It can be seen from Example 1. For example, for $F=\{0, 1\}\in\mathscr{C}\mathscr{S}(X)\subset\mathscr{C}^*\mathscr{S}(X)\subset\mathscr{S}(X)$, $\mathrm{qcl}\{F\}=\mathscr{T}\mathrm{cl}\{F\}\supset H=[0,1]$ and $H\in\mathscr{C}\mathscr{S}(X)\subset\mathscr{C}^*\mathscr{S}(X)\subset\mathscr{S}(X)$.

PROPOSITION 2. *Let* (X, τ^1, τ^2) *be a pairwise* R_1 *space and* $\mathscr{F}$ *be one of the collections* $\mathscr{Q}(X)$ *or* $\mathscr{S}(X)$. *The subcollection* $\tilde{X}$ *is semiclosed in* $(\mathscr{F}, \theta^1, \theta^2)$, *i.e. the collection* $\mathscr{J}_1(\tilde{X})$ *is semiclosed in* $(\bar{\mathscr{F}}, \bar{\theta}^1, \bar{\theta}^2)$, where $\theta^i\supset\underline{\mathscr{T}}^i$ for $i=1,2$.

PROOF. Let $F\in\bar{\mathscr{F}}-\mathscr{J}_1(\tilde{X})$. There are $[x]$, $[y]\in F$ such that $[x]\neq[y]$, moreover $\mathrm{qcl}\{x\}\cap\mathrm{qcl}\{y\}=\phi$. There is an $i\in\{1, 2\}$ such that $x\notin\tau^i\mathrm{cl}\{y\}$. Since X is pairwise R_1, there are disjoint sets $U\in\tau^i$ and $V\in\tau^j$, $j\in\{1, 2\}-\{i\}$, such that $x\in U$, $y\in V$. Then $\mathscr{U}=>U<\in\underline{\mathscr{T}}^i$, $\mathscr{V}=>V<\underline{\mathscr{T}}^j$ and $F\in\mathscr{W}=\mathscr{U}\cap\mathscr{V}$. The collection $\mathscr{W}$ is semiopen in $\mathscr{F}$ and $\mathscr{W}\cap\mathscr{J}_1(\tilde{X})=\phi$.

COROLLARY 5. *Under the assumptions in the previous proposition, the collection* $\tilde{X}$ *is semiclosed in* $\mathscr{C}^*\mathscr{F}$ and its subspace $\mathscr{C}\mathscr{F}$.

The collection $\tilde{X}$ need not be quasiclosed in any of the spaces $\mathscr{Q}(X)$ and $\mathscr{S}(X)$, and their subspaces $\mathscr{C}\mathscr{Q}(X)$ and $\mathscr{C}\mathscr{S}(X)$. It can be seen again from Example 1. The space $(\boldsymbol{R}, \mathscr{L}, \mathscr{D})$ is pairwise regular, but for each $F\in\mathscr{C}\mathscr{Q}(X)-\mathscr{J}_1(\boldsymbol{R})$ and each basic element $\mathscr{B}_i$ for the topology $\mathscr{T}^i$ $(i=1,2)$, $F\in\mathscr{B}_i$ implies $\mathscr{B}_i\cap\mathscr{J}_1(\boldsymbol{R})\neq\phi$. Also, the converse of the above statement do not hold in general, as Example 1 of [7] shows. The space is not pairwise R_1, not even pairwise R_0, but $\tilde{X}$ is semiclosed in $\mathscr{S}(X)=\mathscr{Q}(X)=\mathscr{C}\mathscr{S}(X)=\mathscr{C}\mathscr{Q}(X)$ with the finite topologies.

We emphasize that in general the least upper bound topology $\mathscr{T}$ of $\mathscr{T}^1$ and $\mathscr{T}^2$ is strictly coarser than the finite topology induced by $\tau=\tau^1\vee\tau^2$.

QUESTION. If (X, τ^1, τ^2) is pairwise R_0, $\mathscr{F}$ one of the collections $\mathscr{S}(X)$ or $\mathscr{Q}(X)$ and $\tilde{X}$ is semiclosed in $(\mathscr{F}, \theta^1, \theta^2)$, is X pairwise R_1?

In fact, the proof of Theorem 1 of [3] gives the following improvement of that theorem and Propositions 5 and 6 of [3].

PROPOSITION 3. *Let* $\mathscr{J}_1(X)\subset\mathscr{Q}(X)$ *and* σ *be a collection such that* $\mathscr{Q}(X)\subset\sigma$.
(*i*) *If* $\theta^i\supset\mathscr{T}^i$ *for* $i=1, 2$, *then* (X, τ^1, τ^2) *is quasicompact if and only if* $(\sigma, \theta^1, \theta^2)$ *is quasicompact.*
(*ii*) *If* $\theta^i=\bar{\mathscr{T}}^i$ *and* $\theta^j\supset\underline{\mathscr{T}}^j$ *for* i, $j\in\{1, 2\}$ *and* $i\neq j$, *the quasicompactness of the space* (X, τ^1, τ^2) *implies the quasicompactness of* $(\sigma, \theta^1, \theta^2)$.

From Theorem 1, Proposition 3 and Corollary 2 we have:

PROPOSITION 4. *Let* (X, τ^1, τ^2) *be a pairwise* R_0 *space and* σ *be a collection such that* $\mathscr{Q}(X)\subset\sigma\subset\mathscr{S}(X)$. (*i*) *If* $\theta^i\supset\underline{\mathscr{T}}^i$ *for* $i=1, 2$, *the space* (X, τ^1, τ^2) *is quasicompact if and only if* $(\sigma, \theta^1, \theta^2)$ *is quasicompact.*

(*ii*) *If* $\theta^i=\overline{\mathscr{T}}^i$ *and* $\theta^j\supset\underline{\mathscr{T}}^j$ *for* $i, j\in\{1, 2\}$ *and* $i\neq j$, *the quasicompactness of* (X, τ^1, τ^2) *implies the quasicompactness of the hyperspace* $(\sigma, \theta^1, \theta^2)$.

Let us remark that in Propositions 3 and 4, instead of σ the collection of all quasicompact elements of σ can be considered.

Vasudevan and Goel gave in [10] a detailed investigation of separation axioms in bitopological hyperspaces 2^X, $BC(X)$ and $C(X)$ equipped with finite topologies. The first two are denoted in this paper by $\mathscr{K}(X)$ and $\mathscr{B}(X)$ while the third one is $\mathscr{C}^*\mathscr{K}(X)$. We add here a few statements concerning weak pairwise T_k and pairwise T_k axioms for $k=0, 1, 2$, in the bitopological hyperspaces considered in this paper. We observe that the weak pairwise T_1 axiom is stronger than the pairwise T_1 axiom considered in [10]. (See [7].)

PROPOSITION 5. *For a bitopological space* (X, τ^1, τ^2) *the hyperspace* $(\mathscr{B}(X), \theta^1, \theta^2)$ *is pairwise* T_0, *where* $\theta^i\supset\underline{\mathscr{T}}^i$ *for* $i=1,2$.

PROPOSITION 6. *If* $\sigma\subset\mathscr{Q}(X)$ *and* $\theta^i\supset\underline{\mathscr{T}}^i$, $i=1, 2$, *the hyperspace* $(\sigma, \theta^1, \theta^2)$ *is weak pairwise* T_0.

REMARK. Example 1 shows that Proposition 6 does not hold in general for the collection $\mathscr{S}(X)$, and that the weak pairwise T_0 property cannot be replaced by pairwise T_0. Since $\mathscr{S}(\boldsymbol{R})$ is the collection of all nonempty closed subsets of the real line, $A=\{-1, 1\}$ and $B=\{-1, 0, 1\}$ belong to $\mathscr{C}\mathscr{S}(\boldsymbol{R})\subset\mathscr{S}(\boldsymbol{R})$. For each $V\in\mathscr{L}\cup\mathscr{D}$, $A\in>V<$ if and only if $B\in>V<$ as well as $A\in<V>$ if and only if $B\in<V>$. Hence, $(\mathscr{S}(\boldsymbol{R}), \theta^1, \theta^2)$ and $(\mathscr{C}\mathscr{S}(\boldsymbol{R}), \theta^1, \theta^2)$ are not weak pairwise T_0.

Also, the space $(\mathscr{K}(X), \mathscr{T}^1, \mathscr{T}^2)$ is not pairwise T_0 since for $A=[0, +\infty)$ and $B=\boldsymbol{R}$, there is no $\mathscr{T}^1$ open set $\mathscr{U}$ such that $A\in\mathscr{U}$ and $B\notin\mathscr{U}$, nor a $\mathscr{T}^2$ open set $\mathscr{V}$ such that $A\notin\mathscr{V}$ and $B\in\mathscr{V}$.

Moreover, in Proposition 6 the finite or lower semi-finite topology θ^i cannot be replaced by the upper semi-finite topology $\overline{\mathscr{T}}^i$. It can be seen from the following example.

EXAMPLE 2. Let $X=\boldsymbol{R}$, τ^1 be the left hand topology and τ^2 the indiscrete topology. Then $\mathscr{K}(X)=\{[a, +\infty)\,|\,a\in\boldsymbol{R}\}\cup\{\boldsymbol{R}\}$. In $(\mathscr{K}(X), \mathscr{T}^1, \mathscr{T}^2)$ both topologies are indiscrete, so the space is not weak pairwise T_0.

PROPOSITION 7. *If* (X, τ^1, τ^2) *is pairwise* R_0 *the hyperspace* $(\mathscr{B}(X), \mathscr{T}^1, \mathscr{T}^2)$ *is pairwise* T_1.

COROLLARY 6. *If* (X, τ^1, τ^2) *is pairwise* R_0, *then* $(\mathscr{B}(X), \mathscr{T}^1, \mathscr{T}^2)$ is pairwise R_0.

The converse of the above corollary does not hold in general. The space in Example 1 of [7] is not pairwise R_0, but $\mathscr{B}(X)=\{X\}$, so $(\mathscr{B}(X), \mathscr{T}^1, \mathscr{T}^2)$ is pairwise R_0.

QUESTION. If (X, τ^1, τ^2) is pairwise R_0, is $(\mathscr{F}, \mathscr{T}^1, \mathscr{T}^2)$ pairwise R_0, where $\mathscr{F}$ is $\mathscr{S}(X)$ or $\mathscr{C}(X)$?

But we have:

If (X, τ) is R_0 and $\mathscr{C}(X)$ or $\mathscr{S}(X)$ is pairwise R_0 (pairwise R_1), then (X, τ^1, τ^2) is pairwise R_0 (pairwise R_1).

If (X, τ^1, τ^2) is pairwise R_1, any of the collections $\mathscr{B}(X)$, $\mathscr{K}(X)$, $\mathscr{C}(X)$ and $\mathscr{S}(X)$ with the finite topologies $\mathscr{T}^1$ and $\mathscr{T}^2$ need not be pairwise R_1, as follows from the topological case. (See Theorem 4.9 of [2].)

PROPOSITION 8. *If a bitopological space* (X, τ^1, τ^2) *is pairwise* R_1, *its hyperspace* $(\mathscr{C}^*(X), \mathscr{T}^1, \mathscr{T}^2)$ *is also pairwise* R_1.

PROOF. Let $\mathscr{F}=\mathscr{C}^*(X)$. $F, H\in\mathscr{F}$ and $F\notin\mathscr{T}^i\mathrm{cl}\{H\}$ for $i\in\{1, 2\}$. There is a basic element $\mathscr{W}=\langle W_0\,; W_1, \cdots, W_n\rangle\in\mathscr{T}^i$ such that $F\in\mathscr{W}$ and $H\notin\mathscr{W}$. We consider the following cases:

1. If $H\cap W_k=\phi$ for some $k\in\{1,\cdots,n\}$, then there is an $x\in F\cap W_k$. It follows that $\tau^j\mathrm{cl}\{x\}\subset W_k$ for $j\in\{1, 2\}-\{i\}$. For each $y\in H$, $y\notin\tau^j\mathrm{cl}\{x\}$, so there are $V_y\in\tau^j$, $U_y\in\tau^i$ such that $y\in V_y$, $x\in U_y$ and $U_y\cap U_y=\phi$. A finite subcollection $\{V_{y_1}, \cdots, V_{y_m}\}$ covers H. Let $V=\bigcup_{s=1}^{m} V_{y_s}$ and $U=\bigcap_{s=1}^{m} U_{y_s}$. Then $H\subset V\in\tau^j$, $x\in U\in\tau^i$,

$H\in\mathscr{V}=<V>\in\bar{\mathscr{T}}^j\subset\mathscr{T}^j$, $F\in\mathscr{U}=>U<\in\underline{\mathscr{T}}^i\subset\mathscr{T}^i$ and $\mathscr{U}\cap\mathscr{V}=\phi$.

2. If $H\not\subset W_0$, then there is an $\eta\in H\supset W_0^c$. It follows that $\tau^i\mathrm{cl}\{\eta\}\subset W_0^c$. For each $\varphi\in F$, $F\subset W_0$ implies $\varphi\notin\tau^i\mathrm{cl}\{\eta\}$, so there are $U_\varphi\in\tau^i$, $V_\varphi\in\tau^j$ such that $\varphi\in U_\varphi$, $\eta\in V_\varphi$ and $U_\varphi\cap V_\varphi=\phi$. A finite subcollection $\{U_{\varphi_1}, \cdots, U_{\varphi_m}\}$ covers F. Let $U=\bigcup_{s=1}^m U_{\varphi_s}$ and $V=\bigcap_{s=1}^m V_{\varphi_s}$. Then $F\subset U\in\tau^i$, $\eta\in V\in\tau^j$ and $U\cap V=\phi$. Hence $F\in\mathscr{U}=<U>\in\bar{\mathscr{T}}^i\subset\mathscr{T}^i$, $H\in\mathscr{V}=>V<\in\underline{\mathscr{T}}^j\subset\mathscr{T}^j$ and $\mathscr{U}\cap\mathscr{V}=\phi$.

PROPOSITION 9. *If* (X, τ^1, τ^2) *is pairwise* R_0, *then* $(\mathscr{Q}(X), \mathscr{T}^1, \mathscr{T}^2)$ *is weak pairwise* T_1.

PROOF. Let F, $H\in\mathscr{Q}(X)$ and $F\neq H$. There is an $x\in(F-H)\cup(H-F)$. Let $x\in F-H$. Since $H=H_1\cap H_2$, where H_k is τ^k closed for $k=1, 2$, there is an $i\in\{1,2\}$ such that $x\notin H_i$. X is pairwise R_0, hence $\tau^j\mathrm{cl}\{x\}\cap H_i=\phi$, for $j\in\{1,2\}-\{i\}$. Let $U=H_i^c$ and $V=(\tau^j\mathrm{cl}\{x\})^c$. Then $F\in\mathscr{U}=>U<\in\underline{\mathscr{T}}^i\subset\mathscr{T}^i$, $H\notin\mathscr{U}$, $H\in\mathscr{V}=<V>\in\bar{\mathscr{T}}^j\subset\mathscr{T}^j$ and $F\notin\mathscr{V}$.

From Propositions 8, 9 and G, we have the following result.

COROLLARY 7. *If* (X, τ^1, τ^2) *is pairwise* R_1, *then* $(\mathscr{C}^*\mathscr{Q}(X), \mathscr{T}^1, \mathscr{T}^2)$ and its subspace $(\mathscr{C}\mathscr{Q}(X), \mathscr{T}^1, \mathscr{T}^2)$ *are weak pairwise* T_2.

Theorem 3.3 of [10] can be generalised as follows:

PROPOSITION 10. *If* (X, τ^1, τ^2) *is pairwise regular, then* $(\mathscr{Q}(X), \mathscr{T}^1, \mathscr{T}^2)$ *is weak pairwise* T_2.

Example 1 shows that in Propositions 9 and 10 and Corollary 7, the collection $\mathscr{Q}(X)$ cannot be replaced by $\mathscr{S}(X)$. The space $(\mathscr{S}(\boldsymbol{R}), \mathscr{T}^1, \mathscr{T}^2)$ and its subspace $(\mathscr{C}\mathscr{S}(\boldsymbol{R}), \mathscr{T}^1, \mathscr{T}^2)$ are not even weak pairwise T_0, while $(\boldsymbol{R}, \mathscr{L}, \mathscr{D})$ is pairwise regular. Also, any of the finite topologies cannot be replaced by a semi-finite.

The converse statements to Propositions 9 and 10 and Corollary 7 do not hold, as the following example shows.

EXAMPLE 3. Let $X=\{a, b\}$, $\tau^1=\{\phi, \{a\}, X\}$ and τ^2 be discrete topology. The space (X, τ^1, τ^2) is not pairwise R_0, but the space $(\mathscr{F}, \mathscr{T}^1, \mathscr{T}^2)$ is weak pairwise T_2, where $\mathscr{F}=\mathscr{S}(X)=\mathscr{Q}(X)=\mathscr{K}(X)=\mathscr{A}(X)=\mathscr{C}\mathscr{F}$.

PROPOSITION 11. *If in the bitopological space* (X, τ^1, τ^2) *the space* (X, τ^1) *is* T_1, $\theta^1\supset\bar{\mathscr{T}}^1$ *and* θ^2 *is one of topologies* $\mathscr{T}^2$, $\bar{\mathscr{T}}^2$ *or* $\underline{\mathscr{T}}^2$, *the space* $(\mathscr{A}(X), \theta^1, \theta^2)$ *is weak pairwise* T_0.

COROLLARY 8. *If a space* (X, τ^1, τ^2) *is pairwise* T_1 *and* $\theta^i\supset\bar{\mathscr{T}}^i$ *for* $i=1, 2$, *the hyperspace* $(\mathscr{A}(X), \theta^1, \theta^2)$ *is pairwise* T_0.

The proof of Theorem 2.3 of [10] gives in fact the following improvement.

PROPOSITION 12. *If* (X, τ^1, τ^2) *is pairwise* T_1, $(\mathscr{K}(X), \mathscr{T}^1, \mathscr{T}^2)$ *is weak pairwise* T_1.

PROPOSITION 13. *If* (X, τ^1, τ^2) *is pairwise* T_2, $\theta\supset\bar{\mathscr{T}}^i$ *and* $\theta^j\supset\underline{\mathscr{T}}^j$ *for* $i, j\in\{1, 2\}$ *and* $i\neq j$, *the space* $(\mathscr{C}^i(X), \theta^1, \theta^2)$ *is weak pairwise* T_2.

COROLLARY 9. *If* (X, τ^1, τ^2) *is pairwise* T_2 *and* $\sigma\subset\mathscr{C}^*(X)$, *the space* $(\sigma, \mathscr{T}^1, \mathscr{T}^2)$ *is pairwise* T_2.

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