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NUMERICAL QUADRATURES AND THEIR EFFECT ON THE SOLUTION OF BOUNDARY VALUE PROBLEMS

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1. Introduction

Consider the second order elliptic boundary value problem (Birkhoff [1], Strang and Fix [9]), defined in an open domain Ω with polygonal boundary $\partial\Omega$ by

$$\begin{cases} Lu=-\nabla\cdot(p\nabla u)+qu=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

Under the assumptions p, q are smooth functions and $p\geqslant p_{\min}>0$, $q\geqslant 0$ in Ω, the solution of problem (1) is equivalent to the solution of

$$(Lu,\ v)=(f,\ v) \qquad \text{for all } v\in H_0^1; \tag{2}$$

where H_0^1 is the space of all u in the Sobolev space H^1 with $u=0$ on $\partial\Omega$ and $(\cdot\,,\,\cdot)$ is the inner product defined by $(u,v)=\int_\Omega uv\,du_\Omega$.

The Ritz-Galerkin solution to (2) is to replace H^1 by a finite dimensional subspace S^h contained in H^1. The elements of S^h are of the form $v^h=\sum_{i=1}^{n}\lambda_i\phi_i$, where ϕ_i are basis functions of S^h. In the finite element method, the region Ω is subdivided into a set τ of finite number of disjoint regions, say triangle elements, and the space S^h consists of piecewise polynomials defined on the elements in τ. An approximate solution to (1) can be obtained by solving the following discrete finite element system (Bramble and Zlamal [2], Courant [4])

$$(Lu^h,\ \phi_i)=(f,\ \phi_i) \qquad \text{for all } i=1,2,\cdots,n. \tag{3}$$

In general, the integrals $F_i=(f,\ \phi_i)$ may not be calculated exactly and hence some numerical quadratures will be necessary to approximate these integrals. Herbold and Varga [5] have proposed some quadrature schemes for the numerical solution of a class of boundary value problems by variational technique and investigated the errors introduced in the approximate solutions by such quadrature formulas. In this paper we derive two numerical quadratures for the load vector F_i and analyze the errors introduced by these quadratures.

Error bounds for the quadratures are expressed in term of the Sobolev seminorm in the form $ch^s|f|_{s,\Omega}$. The order s and the constant factor c of the quadrature error are estimated. Experimental results for the two numerical quadrates are discussed in [7].

The barycentric coordinate system, which some engineers call the areal coordinate system, is one of the best local coordinate systems for a study of the finite element method (Ciarlet [3], Strang and Fix [9]). In Section 2 we state the underlying notations and develop some fundamental results of vector calculus in the barycentric coordinate system. In Section 3 two quadrature schemes for calculating the load vector F_i are derived. Quadrature errors and their effect on the finite element solution of boundary value problems using linear splines are analyzed in Section 4. For simplicity, we assume Ω can be triangulated into uniform equilateral triangles and restrict our analysis to the uniform meshs only. The calculation can be generalized to non-uniform meshes through affine transformation.

2. Notation and preliminaries

Denote by τ a triangulation of Ω, by Ω_h the set of all vertices of triangles T in τ, and by $\mathring{\Omega}_h$ the set of all interior nodes of Ω_h.

A linear operator $L_x : \Omega_h \to R$ is called an *q-local operator* for some integer q if the value of $L_x(y)$ at $y \in \Omega_h$ depends only on the values of y in a q-neighbour of x; more precisely, $L_x(y)=0$ for all values of y in Ω_h with $|x-y|>q$, where $|x-y|$ is the minimum number of edges connecting the two nodes x and y along some sides of triangles in τ, (see [8] for further detail).

Let $T=A_0A_1A_2$ be a triangle in a polygonal domain Ω. We will denote by ξ_i, $i=0, 1, 2$ the three affine functions defined by the equation $\xi_i(A_j)=\delta_{ij}$, where δ is the Kronecker delta function. Since A_0, A_1, A_2 are affinely independent, any point $x \in \Omega$ can be uniquely represented as $x=\xi_0(x)A_0+\xi_1(x)A_1+\xi_2(x)A_2$, $\xi_0+\xi_1+\xi_2=1$. We will refer to ξ_0, ξ_1, ξ_2 the barycentric coordinates of x with respect to the triangle $T=A_0A_1A_2$. Any polynomial of degree n on Ω can be expressed uniquely as a homogeneous polynomial of degree n in the barycentric coordinates with respect to a specific triangle $T=A_0A_1A_2$ in τ, or else as a polynomial of degree n in any two of the coordinates. Define the first order linear differential operators D_i with respect to the coordinate ξ_i by

$$D_i(\xi_i)=0 \text{ and } D_i(\xi_{i\pm 1})=\mp 1.$$

That is, the counter clockwise normalized derivative of a function f in the direction parallel to the side opposite A_i.

If f is a differentiable function mapping Ω into R then we have $\sum_i D_i f=0$. The differential operator can be extended to any order by $D^\alpha f=D_0^{\alpha_0}D_1^{\alpha_1}D_2^{\alpha_2}f$, where $\alpha=(\alpha_0, \alpha_1, \alpha_2)\in N^3$, D^0 and D_i^0 denote the identity operator. We denote by $|\alpha|=\alpha_0+\alpha_1+\alpha_2$ the order of D^α. Define $d\mu_T$ and $d\mu_\Omega$ as Lebesgue measure on T and Ω normalized by $\int_T d\mu_T=\int_\Omega d\mu_\Omega=1$. If Ω is a polygon and τ is a triangulation of Ω, then

$$\int_\Omega f\, d\mu_\Omega=\sum_{T\in\tau}\mu_\Omega(T)\int_T f\, d\mu_T$$

In particular, if Ω is an equilateral triangle of unit side length and τ is an equilateral triangulation of Ω, then $\mu_\Omega(T)=h^2$ for all triangles of side length h in τ.

Define dx as Lebesgue measure along the edge $A_{i+1}A_{i-1}$ of a triangle $T=A_0A_1A_2$ normalized by $\int_{A_{i+1}}^{A_{i-1}} dx=1$.

Some results for lines and surfaces integrals are given below.

LEMMA 2.1. *Let* $f: T\to R$, *if* D_i *exists on the side* $A_{i+1}A_{i-1}$, *then*

$$\int_{A_{i+1}}^{A_{i-1}} D_i f(x)\,dx=f(A_{i-1})-f(A_{i+1})$$

LEMMA 2.2. $\int_{A_{i+1}}^{A_{i-1}} gD_i f dx=(gf)(A_{i-1})-(gf)(A_{i+1})-\int_{A_{i+1}}^{A_{i-1}} fD_i g dx$

LEMMA 2.3. $\int_T D_i f d\mu_T=2\int_{A_{i-1}}^{A_i} f dx-2\int_{A_i}^{A_{i+1}} f dx$

LEMMA 2.4. $\int_T gD_i f d\mu_T=2\int_{A_{i-1}}^{A_i} fg dx-2\int_{A_i}^{A_{i+1}} fg dx-\int_T fD_i g d\mu_T$

LEMMA 2.5(Holand Bell [6]).

$$\int_T \xi_0^{s_0}\xi_1^{s_1}\xi_2^{s_2}\, d\mu_T=\frac{s_0!s_1!s_2!2!}{(s_0+s_1+s_2+2)!}$$

Denote by H^k, $k\geqslant 0$ the Sobolev space of real valued functions which together with their generalized derivatives up to the k^{th} order are square integrable over Ω. A norm on a triangle T is given by $\|u\|_{k,T}=\{\sum_{i\leq k}|u|^2_{i,T}\}^{1/2}$, where $|u|_{i,T}=\{\sum_{|\alpha|=i} h^{-2i}\int_T (D^\alpha u)^2 du_T\}^{1/2}$ is the Sobolev semi-norm on the triangle T. The Sobolev norm for H^k can be represented as

$$\|u\|_{k,\Omega}=\{\sum_{T\in\tau}\mu_\Omega(T)\|u\|^2_{k,T}\}^{1/2}.$$

3. Numerical quadrature formulas

For every interior node β of Ω_h, let ϕ_β be the trial function which equals one at β and zero at all other nodes. These pyramid functions ϕ_β form a basis for the trial space S^h.

The basis function $\phi_\beta(x)$ can be expressed as

$$\phi_\beta(x)=\begin{cases}\xi_\beta & \text{if } x\in H_\beta\\ 0 & \text{otherwise}\end{cases}$$

where H_β is the hexagon formed by the six triangles in τ with the common vertex β and ξ_β is the barycentric coordinate of x with respect to any of the triangles in H_β.

In this section, we shall derive two quadratures for $F_\beta=\int_\Omega f\phi_\beta d\mu_\Omega$. The first one is a 0-local operator while the second one is a 1-local operator. A 0-local quadrature $\tilde{F}_\beta$ for the integral F_β is given by $\tilde{F}_\beta=af(\beta)$. The constant a can be determined as follows:

$$\int_\Omega \phi_\beta d\mu_\Omega - a=0.$$

It follows that

$$a=\int_\Omega \phi_\beta d\mu_\Omega=6\mu_\Omega(T)\int_T \xi_\beta d\mu_T=2\mu_\Omega(T).$$

It is easy to verify that the 0-local quadrature is exact for all polynomials of degrees less than or equal to 1.

To obtain numerical quadrature with higher order of accuracy, we require the following lemma:

LEMMA 3.1. *Let x_j, $j=1,2,\cdots,6$ be the six vertices of a hexagon H_{x_0} and ψ be a quadratic polynomial which takes the value* 1 *along the edges x_6x_1 and x_3x_4 and vanishes along the line x_2x_5 of the hexagon H_{x_0}. Then*

$$\int_\Omega \psi\phi_{x_0} d\mu_\Omega=\frac{\mu_\Omega(T)}{3}$$

PROOF. Denote the barycentric coordinates of a point $x \in H_{x_0}$ with respect to the triangle $T_j = x_0 x_j x_{j+1}$ by (ξ, η, κ). Then the polynomial ψ can be represented in terms of the barycentric coordinates of x with respect to the six triangles T_j as follows:

$$\psi(\xi, n, \kappa) = \begin{cases} \kappa^2 & \text{in } T_2 \text{ and } T_5 \\ \eta^2 + 2\eta\kappa + \kappa^2 & \text{in } T_3 \text{ and } T_6 \\ \eta^2 & \text{in } T_1 \text{ and } T_4. \end{cases}$$

It follows that

$$\begin{aligned} \int_\Omega \psi\phi_\beta d\mu_\Omega &= 4\mu_\Omega(T) \int_T \xi(\eta^2 + \kappa^2 + \eta\kappa) d\mu_T \\ &= 4\mu_\Omega(T)\left(\frac{2!2!}{5!} + \frac{2!2!}{5!} + \frac{2!}{5!}\right) \\ &= \frac{\mu_\Omega(T)}{3}. \end{aligned}$$

From the result of Lemma 3.1, we observe that the 0-local numerical quadratures $\tilde{F}_\beta = 2\mu_\Omega(T)$ is not exact for all polynomials of degree 2. Now we shall derive another quadrature which is exact for polynomials of higher degree.

Consider the 1-local quadrature $\tilde{F}_\beta = af(\beta) + b \sum_{|\beta-\gamma|=1} f(\gamma)$, the two parameters a and b can be determined as follows:

$$\begin{cases} \int_\Omega \phi_\beta d\mu_\Omega - a - 6b = 0 \\ \int_\Omega \psi\phi_\beta d\mu_\Omega - 4b = 0. \end{cases}$$

It follows from Lemma 3.1 that $b = \frac{\mu_\Omega(T)}{12}$ and $a = \frac{3}{2}\mu_\Omega(T)$.

4. Quadrature errors and their effect on the numerical solution of boundary value problems

In this section, we analyze the errors of the two quadratures derived in Section 3 and discuss the effect of quadrature errors to the finite element solution of the boundary value problems using linear splines.

For each $\beta \in \mathring{\Omega}_h$ let $E(f, \beta)$ denote the quadrature error $F_\beta - \tilde{F}_\beta$.

For simplicity, we denote by x_0 the centre β of a hexagon H_β and by x_j, $j = 1, \cdots, 6$ the six vertices of H_β.

THEOREM 4.1. *If* $f\in H^2$ *and* $\tilde{F}_\beta(f)=2\mu_\Omega(T)\,f(\beta)$, *then*

$$\Big(\sum_{\beta\in\Omega_h}|E(f,\ \beta)|^2\Big)^{1/2}\leqslant\Big[\frac{23}{560}\mu_\Omega(T)\Big]^{1/2}h^2|f|_{2,\Omega}.$$

To prove the theorem, we need the following two auxiliary lemmas.

LEMMA 4.1. *If* $f\in H^2$ *and* $P(\xi_0)$ *is a real valued function of* ξ_0 *defined on each of the triangular elements* $T_j=x_0x_jx_{j+1}$, *then*

$$\sum_{j=1}^{6}\int_{x_j}^{x_0}P(\xi_0)D_{x_jx_0}f\,dx=-\sum_{j=1}^{6}\int_{T_j}\frac{1}{2}P(\xi_0)D_{00}f\,d\mu_{T_j}$$

PROOF. The result of the lemma follows from the decomposition of the derivatives $D_{x_jx_0}f$ into the sum of the two derivates D_0f and D_1f.

LEMMA 4.2. *If* $f\in H^2$, *then the error of the* 0-*local numerical quadrature has a representation of the form*:

$$E(f,\ \beta)=\int_\Omega f\phi_\beta d\mu_\Omega-2h^2f(\beta)$$
$$=\sum_{j=1}^{6}\mu_\Omega(T)\int_{T_j}\Big(\frac{\xi_0^2}{2}-\frac{\xi_0^3}{3}-\frac{\xi_0\xi_1\xi_2}{2}\Big)D_{00}f\,d\mu_{T_j}$$

PROOF. $E(f,\ \beta)=\int_\Omega f\phi_\beta d\mu_\Omega-2h^2f(\beta)$

$$=\sum_{j=1}^{6}\mu_\Omega(T)\Big[\int_{T_j}f\xi_0d\mu_{T_j}-\int_{T_j}f(x_0)\xi_0d\mu_{T_j}\Big]$$
$$=\sum_{j=1}^{6}\mu_\Omega(T)\int_{T_j}[f-f(\beta)]\xi_0d\mu_{T_j}$$
$$=\sum_{j=1}^{6}\mu_\Omega(T)\int_{T_j}[f-f(\beta)]\Big[D_0\frac{\xi_0}{2}(\xi_2-\xi_1)\Big]d\mu_{T_j}$$

It follows from Lemma 2.4 and the fact ξ_1 and ξ_2 vanish on $x_{j+1}x_0$ and x_0x_j respectively that

$$E(f,\ \beta)=\sum_{j=1}^{6}\mu_\Omega(T)\Big[\int_{x_{j+1}}^{x_0}\xi_0(1-\xi_0)(f-f(\beta))dx$$
$$+\int_{x_0}^{x_j}\xi_0(1-\xi_0)(f-f(\beta))dx+\int_{T_j}D_0\Big(\frac{1}{2}\xi_0\xi_1\xi_2\Big)D_0fd\mu_{T_j}\Big]$$
$$=\sum_{j=1}^{6}\mu_\Omega(T)\Big[2\int_{x_{j+1}}^{x_0}\xi_0(1-\xi_0)(f-f(\beta))dx+\int_{x_{j+1}}^{x}\xi_0\xi_1\xi_2D_0fdx$$
$$-\int_{x_0}^{x_j}\xi_0\xi_1\xi_2D_0fdx-\int_{T_j}\frac{1}{2}\xi_0\xi_1\xi_2D_{00}fd\mu_{T_j}\Big]$$

$$= \sum_{j=1}^{6} \mu_\Omega(T)\Big[2\Big(\frac{\xi_0^2}{2}-\frac{\xi_0^3}{3}\Big)(f-f(\beta))\Big|_{x_j}^{x_0} - 2\int_{x_j}^{x_0}\Big(\frac{\xi_0^2}{2}-\frac{\xi_0^3}{3}\Big)D_{x_jx_0} f dx$$

$$-\int_{T_j}\frac{1}{2}\xi_0\xi_1\xi_2 D_{00} f d\mu_{T_j}\Big]$$

$$= -\mu_\Omega(T)\sum_{j=1}^{6}\Big[\int_{x_j}^{x_0}\Big(\xi_0^2-\frac{2}{3}\xi_0^3\Big)D_{x_jx_0} f dx + \int_{T_j}\frac{1}{2}\xi_0\xi_1\xi_2 D_{00} f d\mu_{T_j}\Big]$$

It follows from Lemma 4.1 that the above equation gives Lemma 4.2.

PROOF OF THEOREM 4.1. Application of the triangle inequality and the Cauchy-Schwarz inequality to the equation in Lemma 4.2, we get

$$|E(f,\ \beta)| \leqslant \sum_{j=1}^{6}\mu_\Omega(T)\Big[\int_{T_j}\Big(\frac{1}{2}\xi_0^2-\frac{1}{3}\xi_0^2-\frac{1}{2}\xi_0\xi_1\xi_2\Big)^2 d\mu_{T_j}\Big]^{1/2}\|D_{00} f\|_{L^2(T_j)}$$

$$= \Big(\frac{23}{3360}\Big)^{1/2}\mu_\Omega(T)\sum_{j=1}^{6}\|D_{00} f\|_{L^2(T_j)}$$

It follows that

$$\sum_{\beta\in\Omega_h^\circ} |E(f,\ \beta)|^2 \leqslant \frac{23}{3360}\mu_\Omega(T)\sum_{\beta\in\Omega_h^\circ}\mu_\Omega(T)\Big[\sum_{j=1}^{6}\|D_{00} f\|_{L^2(T_j)}\Big]^2$$

$$\leqslant \frac{23}{560}\mu_\Omega(T)\sum_{\beta\in\Omega_h^\circ}\mu_\Omega(T)\Big[\sum_{j=1}^{6}\|D_{00} f\|^2_{L^2(T_j)}\Big]$$

We observe that for each $T=x_0x_1x_2\in\tau$ and each i, the term $\|D_{ii}\ f\|^2_{L^2(T_j)}$ appears in the right hand side of the above inequality at most once; thus we have

$$\Big(\sum_{\beta\in\Omega_h^\circ}|E(f,\ \beta)|^2\Big)^{1/2} \leqslant \Big[\frac{23}{560}\mu_\Omega(T)\Big]^{1/2}\Big[\sum_{T\in\tau}\mu_\Omega(T)\sum_{i=1}^{3}\|D_{ii} f\|^2_{L^2(T)}\Big]^{1/2}$$

$$\leqslant \Big[\frac{23}{560}\mu_\Omega(T)\Big]^{1/2} h^2 |f|_{2,\Omega}.$$

LEMMA 4.2. *If* $f\in H^4$ *and* $\tilde{F}_\beta$ *is the* 1-*local numerical quadrature of* F_β, *then*

$$\Big(\sum_{\beta\in\Omega_h^\circ}|E(f,\ \beta)^2\Big)^{1/2} \leqslant 0.07208(\mu_\Omega(T))^{1/2} h^4 |f|_{4,\Omega}$$

To prove the theorem, we need the following auxiliary lemmas.

LEMMA 4.3. *If* $f\in H^4$ *and* $P(\xi_0)$ *is a real valued function of* ξ_0 *defined on each of the triangular elements* $T_j=x_0x_jx_{j+1}$, *then,*

$$\sum_{j=1}^{6}\Big[\int_{x_0}^{x_j}P(\xi_0)D_{200} f dx - \int_{x_{j+1}}^{x_0}P(\xi_0)D_{100} f dx\Big]$$

$$=\sum_{j=1}^{6}\int_{T_j}\frac{1}{2}P(\xi_0)(D_{0000}f-\frac{1}{2}D_{0012}f)d\mu_{T_j}$$

PROOF. $\sum_{j=1}^{6}\Big[\int_{x_0}^{x_j}P(\xi_0)D_{200}fdx-\int_{x_{j+1}}^{x_0}P(\xi_0)D_{100}fdx\Big]$

$$=\sum_{j=1}^{6}\Big[\int_{x_0}^{x_j}P(\xi_0)(-D_{000}f-D_{100}f)dx-\int_{x_{j+1}}^{x_0}P(\xi_0)(-D_{000}f-D_{200}f)dx\Big]$$

$$=\sum_{j=1}^{6}\Big[\int_{x_{j+1}}^{x_0}P(\xi_0)D_{000}fdx-\int_{x_0}^{x_j}P(\xi_0)D_{000}fdx$$
$$+\int_{x_{j+1}}^{x_0}P(\xi_0)D_{200}fdx-\int_{x_0}^{x_j}P(\xi_0)D_{100}fdx\Big]$$

$$=\sum_{j=1}^{6}\Big[\int_{T_j}\frac{1}{2}P(\xi_0)D_{0000}fd\mu_{T_j}+\int_{x_{j+1}}^{x}P(\xi_0)D_{200}fdx-\int_{x_0}^{x_j}P(\xi_0)D_{100}fdx\Big]$$

We note that the derivative $-D_{100}f$ along x_jx_0 with respect to T_j is the same as $D_{220}f$ with respect to T_{j-1}, and $D_{200}f$ along $x_{j+1}x_0$ with respect to T_j is the same as $-D_{110}f$ with respect to T_{j+1}. Thus, these derivatives can be divided into two equal parts, half of them will be added to the line integral of the adjacent triangle. It follows that

$$\sum_{j=1}^{6}\Big[\int_{x_0}^{x_j}P(\xi_0)D_{200}fdx-\int_{x_{j+1}}^{x_0}P(\xi_0)D_{100}fdx\Big]$$

$$=\sum_{i=1}^{6}\Big[\int_{T_j}\frac{1}{2}P(\xi_0)D_{0000}fd\mu_{T_j}+\int_{x_{j+1}}^{x_0}\frac{1}{2}P(\xi_0)(D_{200}f+D_{220}f)dx$$
$$-\int_{x_0}^{x_j}\frac{1}{2}P(\xi_0)(D_{100}f+D_{110}f)dx\Big]$$

$$=\sum_{j=1}^{6}\Big[\int_{T_j}\frac{1}{2}P(\xi_0)D_{0000}fd\mu_{T_j}-\int_{x_{j+1}}^{x_0}\frac{1}{2}P(\xi_0)D_{210}fdx$$
$$+\int_{x_0}^{x_j}\frac{1}{2}P(\xi_0)D_{120}fdx\Big]$$

which gives Lemma 4.3.

LEMMA 4.4. *If $f\in H^4$, then the error functional of the* 1-*local numerical quadrature has a representation of the form*

$$E(f,\ \beta)=\int_{\Omega}f\phi_\beta d\mu_\Omega-\mu_\Omega(T)\Big[\frac{3}{2}f(\beta)+\frac{1}{12}\sum_{j=1}^{6}f(x_j)\Big]$$
$$=\frac{1}{24}\mu_\Omega(T)\sum_{j=1}^{6}\int_{T_j}\Big\{\Big[\Big(\frac{1}{2}+\xi_0-8\xi_0^2+5\xi_0^3+\xi_0\xi_1\xi_2\Big)\xi_1\xi_2$$

$$-\frac{\xi_0}{2}-\frac{\xi_0^2}{4}+3\xi_0^3-\frac{13}{4}\xi_0^4+\xi_0^5\Big]D_{0000}f$$

$$+\frac{1}{2}\Big(\frac{\xi_0}{2}+\frac{\xi_0^2}{4}-3\xi_0^3+\frac{13}{4}\xi_0^4-\xi_0^5\Big)D_{0012}f\Big\}d\mu_{T_j}$$

PROOF. We shall only give a brief proof for this lemma.

$$E(f,\ \beta)=\Big[\int_{\Omega}f\phi_\beta d\mu_\Omega-2h^2f(\beta)\Big]+\frac{\mu_\Omega(T)}{12}\sum_{j=1}^{6}[f(\beta)-f(x_j)]$$

$$=\Big[\int_{\Omega}f\phi_\beta d\mu_\Omega-2h^2f(\beta)\Big]+\frac{1}{12}\mu_\Omega(T)\sum_{j=1}^{6}\int_{x_j}^{x_0}D_{x_jx_0}fdx$$

It follows from Lemma 4.2 and Lemma 4.1 that

$$E(f,\ \beta)=\sum_{j=1}^{6}\frac{\mu_\Omega(T)}{24}\int_{T_j}(-1+12\xi_0^2-8\xi_0^3-12\xi_0\xi_1\xi_2)D_{00}fd\mu_{T_j}$$

which reduces to the following result

$$E(f,\ \beta)=\frac{\mu_\Omega(T)}{24}\sum_{j=1}^{6}\Big[\int_{x_{j+1}}^{x_0}(-1-2\xi_0+16\xi_0^2-10\xi_0^3)\xi_2D_{00}fdx$$

$$+\int_{x_0}^{x_j}(-1-2\xi_0+16\xi_0^2-10\xi_0^3)\xi_1D_{00}fdx$$

$$-\int_{T_j}\frac{1}{2}(-1-2\xi_0+16\xi_0^2-10\xi_0^3)(\xi_2-\xi_1)D_{000}fd\mu_{T_j}$$

$$-\int_{T_j}D_0(\xi_0\xi_1^2\xi_2^2)D_{000}fd\mu_{T_j}\Big]$$

After further evaluation, we have

$$E(f,\ \beta)=\frac{\mu_\Omega(T)}{24}\sum_{j=1}^{6}\Big[\int_{x_0}^{x_j}\Big(-\xi_0-\frac{\xi_0^2}{2}+6\xi_0^3-\frac{13}{2}\ \xi_0^4+2\xi_0^5\Big)D_{200}f\,dx$$

$$-\int_{x_{j+1}}^{x_0}\Big(-\xi_0-\frac{\xi_0^2}{2}+6\xi_0^3-\frac{13}{2}\ \xi_0^4+2\xi_0^5\Big)D_{100}f\,dx$$

$$+\int_{T_j}\Big(\frac{1}{2}+\xi_0-8\xi_0^2+5\xi_0^3+\xi_0\xi_1\xi_2\Big)\xi_1\xi_2D_{0000}f\,d\mu_{T_j}$$

It follows from Lemma 4.3 that the above equation gives Lemma 4.4.

PROOF OF THEOREM 4.2. Applying the triangle inequality and the *Cauchy-Schwarz inequality* to the equation in Lemma 4.4, we get

$$|E(f,\ \beta)|\leqslant\frac{\mu_\Omega(T)}{24}\sum_{j=1}^{6}(c_0\|D_{0000}f\|_{L^2(T_j)}+c_1\|D_{0012}f\|_{L^2(T_j)})$$

where $c_0=0.08495$ and $c_1=0.04297$. Applying the Cauchy-Schwarz inequality to the above inequality, we get

$$|E(f,\ \beta)|\leqslant\frac{\mu_\Omega(T)}{24}\left(6c_0^2+6c_1^2\right)^{1/2}\{\sum_{j=1}^{6}(\|D_{0000}f\|^2_{L^2(T_j)}+\|D_{0012}f\|^2_{L^2(T_j)})\}^{1/2}$$

It follows that

$$(\sum_{\beta\in\mathring{\Omega}_h}|E(f,\beta)|^2)^{1/2}\leqslant 0.07208(\mu_\Omega(T))^{1/2}\{\sum_{\beta\in\mathring{\Omega}_h}\mu_\Omega(T)\sum_{j=1}^{6}(\|D_{0000}f\|^2_{L^2(T_j)}$$
$$+\|D_{0012}f\|^2_{L^2(T_j)}\}^{1/2}$$

for each $T\in\tau$, since the terms $\|D_{0000}f\|^2_{L^2(T_j)}$ and $\|D_{0012}f\|^2_{L^2(T_j)}$ appear in the right hand side of the above inequality at most once, thus, we have

$$(\sum_{\beta\in\mathring{\Omega}_h}|E(f,\beta)|^2)^{1/2}\leqslant 0.07208(\mu_\Omega(T))^{1/2}(\sum_{T\in\tau}\mu_\Omega(T)\sum_{|\alpha|=4}\|D^\alpha f\|^2_{L^2(T)})^{1/2}$$
$$=0.07208(\mu_\Omega(T))^{1/2}h^4|f|_{4,\Omega}.$$

If the load vector $F_\beta=(f,\phi_\beta)$ of (3) is approximated by a numerical quadrature $\tilde{F}_\beta$, then we are solving

$$(\tilde{u}^h,\ \phi_\beta)=\tilde{F}_\beta,\ \ \beta\in\mathring{\Omega}_h \tag{4}$$

where $\tilde{u}^h=\sum_{\beta\in\mathring{\Omega}_h}\tilde{\lambda}_\beta\phi_\beta$ is a solution to the linear system (4).

From (4), we get

$$(u^h-\tilde{u}^h,\ \phi_\beta)=F_\beta-\tilde{F}_\beta=E(F,\ \beta)$$

It follows that

$$(u^h-\tilde{u}^h,\ u^h-\tilde{u}^h)\sum_{\beta\in\mathring{\Omega}_h}(\lambda_\beta-\tilde{\lambda}_\beta)E(f,\ \beta)$$

which reduces to

$$\|u^h-\tilde{u}^h\|_a^2\leqslant\sum_{\beta\in\mathring{\Omega}_h}|\lambda_\beta-\tilde{\lambda}_\beta||E(f,\ \beta)|$$

By applying the Cauchy-Schwarz inequality to the right hand side of the above inequality, we get

$$\|u^h-\tilde{u}^h\|_a^2\leqslant(\sum_{\beta\in\mathring{\Omega}_h}(\lambda_\beta-\tilde{\lambda}_\beta)^2)^{1/2}(\sum_{\beta\in\mathring{\Omega}_h}|E(f,\ \beta)|^2)^{1/2}. \tag{5}$$

To obtain an upper bound for $(\sum_{\beta\in\mathring{\Omega}_h}(\lambda_\beta-\tilde{\lambda}_\beta)^2)^{1/2}$ in terms of the L^2 norm $\|u^h-\tilde{u}^h\|_{L^2(\Omega)}$, we get the following lemma.

LEMMA 4.5. *If* $u(x)=\sum_{\beta\in\mathring{\Omega}_h}\lambda_\beta\phi_\beta(x)$ vanishes on $\partial\Omega$, then

$$\|u\|^2_{L^2(\Omega)} \geqslant \frac{1}{2}\mu_\Omega(T) \sum_{\beta\in\mathring{\Omega}_h} \lambda_\beta^2$$

PROOF. $\|u\|^2_{L^2(\Omega)} = \int_\Omega u^2 d\mu_\Omega$

$$= \mu_\Omega(T) \sum_{T\in\tau} \int_T (\lambda_\beta\phi_\beta + \lambda_\gamma\phi_\gamma + \lambda_\kappa\phi_\kappa)^2 d\mu_T$$

$$= \frac{1}{6}\mu_\Omega(T) \sum_{T\in\tau} (\lambda_\beta^2 + \lambda_\gamma^2 + \lambda_\kappa^2 + \lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\kappa + \lambda_\kappa\lambda_\beta)$$

where $\lambda_\beta, \lambda_\gamma, \lambda_\kappa$ are the values of u at the three vertices of $T=\beta\gamma\kappa$.

Since $\lambda_\beta^2+\lambda_\gamma^2+\lambda_\kappa^2+2(\lambda_\beta\lambda_\gamma+\lambda_\gamma\lambda_\kappa+\lambda_\kappa\lambda_\beta)\geqslant 0$ for all $\lambda_\beta, \lambda_\gamma, \lambda_\kappa \in R$, we have

$$\sum_{T\in\tau} (\lambda_\beta^2+\lambda_\gamma^2+\lambda_\kappa^2+\lambda_\beta\lambda_\gamma+\lambda_\gamma\lambda_\kappa+\lambda_\kappa\lambda_\beta) \geqslant \sum_{T\in\tau} \frac{1}{2}(\lambda_\beta^2+\lambda_\gamma^2+\lambda_\kappa^2) \tag{6}$$

Since $\lambda_\beta=0$ for all $\beta\in\partial\Omega$ and for each $\beta\in\mathring{\Omega}_h$, there are six triangles T in τ with the common vertex β. Thus the right hand side of the inequality (6) can be written as $3\sum_{\beta\in\mathring{\Omega}_h}\lambda_\beta^2$. It follows that

$$\|u\|^2_{L^2(\Omega)} \geqslant \frac{1}{2}\mu_\Omega(T) \sum_{\beta\in\mathring{\Omega}_h} \lambda_\beta^2.$$

Since $u^h-\tilde{u}^h$ is a piecewise linear function on Ω and vanishes on the boundary $\partial\Omega$, we can apply Lemma 4.5 to inequality (5) to get

$$\|u^h-\tilde{u}^h\|_a^2 \leqslant \left(\frac{2}{\mu_\Omega(T)}\right)^{1/2} \|u^h-\tilde{u}^h\|_{L^2(\Omega)} \left(\sum_{\beta\in\mathring{\Omega}_h} |E(f,\beta)|^2\right)^{1/2}.$$

Since the energy norm $\|\cdot\|_a$ is equivalent to the Sobolev norm $\|\cdot\|_{1,\Omega}$; we have

$$\|u^h-\tilde{u}^h\|_a \geqslant c\|u^h-\tilde{u}^h\|_{1,\Omega} \geqslant c\|u^h-\tilde{u}^h\|_{L^2(\Omega)}$$

for some constant $c>0$. It follows that

$$\|u^h-\tilde{u}^h\|_a \leqslant \frac{\sqrt{2}}{c(\mu_\Omega(T))^{1/2}} \left(\sum_{\beta\in\mathring{\Omega}_h} |E(f,\beta)|^2\right)^{1/2}$$

If the 0-local numerical quadrature is used, from Theorem 4.1, we have

$$\|u^h-\tilde{u}^h\|_a \leqslant \left(\frac{23}{280}\right)^{1/2} \frac{1}{c} h^2 |f|_{2,\Omega}$$

and if the 1-local numerical quadrature is used, from Theorem 4.2, we have

$$\|u^h-\tilde{u}^h\|_a \leqslant \frac{0.1019}{c} h^4 |f|_{4,\Omega}$$

If u is the exact solution to $Lu=f$, from the triangle inequality, we get

$$||u-\tilde{u}^h||_a \leq ||u-\tilde{u}^h||_a + ||u^h-\tilde{u}^h||_a$$

Since the energy norm $||u-\tilde{u}^h||_a$ has an order of accuracy $0(h^2)$ [9], and the energy norm $||u^h-\tilde{u}^h||_a$ has an order of accuracy $0(h^2)$ and $0(h^4)$ for the 0-local and 1-local numerical quadrature respectively, both numerical quadratures are consistent in the energy norm. That is, the solution still has an order of accuracy $0(h^2)$ in the energy norm for the 0-local and 1-local numerical quadratures.

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