Kyungpook Math. J.
Volume 25, Number 1
June, 1985

ON THE SIMPLICITY OF REAL ROOTS OF REAL POLYNOMIALS

By Uri Leyson

Introduction: T. Craven and G. Csordas proved

THEOREM 1. ([1, Theorem 2.3]). *Given* $h(x)=\sum_{k=0}^{n} b_k x^k$, $n\geq 1$, $b_0\neq 0$, $b_n=1$, *with only real roots and* $f(y)$ *an arbitrary polynomial, form* $F(x, y)=\sum_{k=0}^{n} b_k x^k f^{(k)}(y)$. *Each branch of the real curve* $F(x, y)=0$ *which intersects the y-axis will intersect the y-axis in exactly one point and will intersect each vertical line* $x=c$, *where c is an arbitrary constant. If* $b_0=0$, *the conclusion still holds for all branches which do not coincide with the y-axis. Furthermore, if two branches which cross the y-axis intersect at a singular point* (x_0, y_0) *not on the y-axis, then these branches are in fact components of the form* $y-y_0=0$, *and thus coincide as horizontal lines.*

Throughout the present paper all the polynomials are supposed to have only real coefficients. When we speak about "curve", we always mean the real part of an algebraic curve. In particular $F(x, y)=0$ denotes the curve in Theorem 1. The meaning of the word "branch" is the same as in [1].

In Theorem 2 (a) we find sufficient conditions so that the polynomial $\sum_{k=0}^{n} b_k f^{(k)}(x)$ has at least as many distinct real roots as f has. If, in addition to the above conditions, f has only real roots, then $\sum_{k=0}^{n} b_k f^{(k)}(x)$ has only real simple roots. This is the result of Theorem 2(b). Then we get Corollary 1 which has special meaning in the light of [5]. In Theorem 3(a) and (b), we apply to the polynomials $\sum_{k=0}^{m} k!\, a_k b_k x^k$ and $\sum_{k=0}^{m} b_k x^k f^{(k)}(x)$ the same method as we have used for $\sum_{k=0}^{n} b_k f^{(k)}(x)$ in Theorem 2(a) and (b). This generalizes [3, Theorem 6, pp. 336-337].

In Theorem 3(c), we introduce a new curve having the same construction

as $F(x, y)=0$. Moreover, its construction is invariant under certain perturbation.

Corollary 2 of Theorem 3 is very similar to the result of [2].

THEOREM 2. *Let* $h(x)=\sum_{k=0}^{n} b_k x^k$ *be a polynomial with only real roots,* $b_0 b_m \neq 0$ *and let* $f(x)=\sum_{k=0}^{m} a_k x^k$; $(m\leq n)$ *be a polynomial with r real roots; then: (a) The polynomial* $g(x)=\sum_{k=0}^{n} b_k f^{(k)}(x)$ *has at least r distance real roots.*

(b) If f has only real roots, then g has only real simple roots.

PROOF (b). $b_m \neq 0$ and $m\leq n$ so $\deg(f)\leq\deg(h)$; thus,

$$\lim_{|x|\to\infty} \frac{F(x, tx)}{a_m x^m}=b_0 t^m+b_1 m t^{m-1}+\cdots+b_m m!=p(t) \quad (*)$$

(See [1, Theorem 3.1]) and the real roots of $p(t)$ are the slopes of the asymptotoes of the branches of $F(x, y)=0$. If we assume that g has a root c of multiplicity $2\leq s$, then $F(x, y)=0$ has s branches intersecting the y-axis and meeting at the point $(1, c)$. (See [1, Corollary 3.4]) From Theorem 1 it follows that these s branches coincide as horizontal lines. Therefore, $F(x, y)=0$ has asymptotes with zero slope, i.e. $p(0)=0$. On the other hand, if we put $t=0$ in (*), we get $p(0)=b_m m!$ in contradiction to the conditions $b_m \neq 0$ and $0\leq m$. This contradiction completes the proof of (b).

PROOF (a). Like in (b), (*) holds, and also $F(x, y)=0$ has r branches which intersect the y-axis. Thus by Theorem 1. These r branches intersect the line $x=1$. If we assume that two or more of the above branches meet at a point $(1, c)$, then like in (b) we get $b_m=0$ in contradiction to the assumption of the Theorem.

COROLLARY 1. *If we choose, in Theorem 2 (b), an* $\varepsilon\geq 0$ *which is small enough, then for every choice of* $0\leq\varepsilon_k\leq\varepsilon$, $k=0, 1, 2, \cdots, n$ *the polynomial* $\sum_{k=0}^{n} (b_k+\varepsilon_k) f^{(k)}(x)$ *has only real roots. This result is significant in the light of* [5].

EXAMPLE 1. The polynomials $h(x)=\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n$ and $f(x)=x^m$ $(2\leq m\leq n)$ satisfy the conditions of Theorem 2(b). So the polynomial $g(x)=\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)$ $=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}k!x^{m-k}$ has only real simple roots although the polynomial f

has only one multiple root at $x=0$. In this case we can check it in the following way. If we assume that g has a root c of multiplicity $2\leq s$, then from [1, Corollary 3.4], it follows that $c=0$ (note: $2\leq s\leq m$) so $g(0)=0$ but $g(0)=\binom{n}{m}m!\neq 0$.

THEOREM 3. *Let* $h(x)=\sum_{k=0}^{n} b_k x^k$ *be a polynomial with only negative roots and let* $f(x)=\sum_{k=0}^{m} a_k x^k$ $(m=\deg(f)\leq\deg(h)=n)$ *be a polynomial with r negative roots, s positive roots and u roots at* $x=0$. *Then*:

(a) *Each of the polynomials* $\sum_{k=0}^{m} k!\, a_k b_k x^k$ *and* $\sum_{k=0}^{m} b_k x^k f^{(k)}(x)$ *has at least r negative distinct roots, at least s positive distinct roots and exactly u roots at* $x=0$.

(b) *If f has only real roots, then each of the polynomials* $\sum_{k=0}^{m} k!\, a_k b_k x^k$ *and* $\sum_{k=0}^{m} b_k x^k f^{(k)}(x)$ *has exactly u roots at* $x=0$ *and all their other roots are real and simple.*

(c) *If in addition to the condition (b) we assume* $f(0)\neq 0$, *then there is such a small* $\varepsilon\geq 0$ *that for any choice of* $0\leq\varepsilon_k\leq\varepsilon$ $k=0,1,2,\cdots,n$ *the curve* $\sum_{k=0}^{n}(b_k+\varepsilon_k)x^k f^{(k)}(0) f^{(k)}(y)=0$ *has the same construction as* $F(x, y)=0$.

PROOF (b). f has only real roots. As in [1, Corollary 3,4] $F(x, y)=0$ has only branches which intersect the y-axis. Like in Theorem 2, (*) holds. If we assume that $F(x, 0)=\sum_{k=0}^{m} k!\, a_k b_k x^k$ has a root $c\neq 0$ of multiplicity $2\leq s'$, then it follows that $F(x, y)=0$ has s' branches which intersect the y-axis and meet at the point $(c, 0)$, $0\neq c$. By Theorem 1, these s' branches coincide with the x-axis i.e. $p(0)=0$. On the other hand, if we put $t=0$ in (*), we get $p(0)=m!\, b_m$ so $b_m=0$. Since h has only negative roots, $b_k>0$ for every $k=0, 1, 2, \cdots, n$ in particular $b_m>0$. Therefore we obtain a contradiction. This completes the proof of Theorem 3(b) related to $\sum_{k=0}^{m} k!\, a_k b_k x^k$. The proof of Theorem 3 (b) related to $\sum_{k=0}^{m} b_k x^k f^{(k)}(x)$ is similar to the above.

PROOF (a). Corresponding to every negative root of f, $F(x, y)=0$ has a branch intersecting the negative part of the y-axis and the negative part of the x-axis. (See [1, Corollary 3.6]). Like in the proof of Theorem 3 (b),

no pair among these r branches intersects each other on the x-axis. Thus $F(x, 0)$ has at least r distinct negative roots. In the same way, we prove that $F(x, 0)$ has at least s distinct positive roots. Also it is clear that $F(x, 0)$ has exactly u roots at $x=0$. This completes the proof of Theorem 3 (a) related to $\sum_{k=0}^{m} k!\ a_k b_k x^k$. The proof of Theorem 3 (a) related to $\sum_{k=0}^{m} b_k x^k f^{(k)}(x)$ can be done in a similar way.

PROOF (c). The polynomial $\sum_{k=0}^{m} k!\ a_k b_k x^k$ has only real simple roots. Therefore if we choose an $\varepsilon \geq 0$ which is small enough, then for every choice of $0 \leq \varepsilon_k \leq \varepsilon$ $k=0, 1, 2, \cdots, n$ the polynomial $\sum_{k=0}^{m} k!\ a_k (b_k+\varepsilon_k) x^k$ has only real roots whose number is m. Applying Theorem 1 to $f(y)$ and $\sum_{k=0}^{m} k!\ a_k (b_k+\varepsilon_k) x^k$ completes the proof of (c).

REMARK. If in Theorem 3 h has only positive roots, then the results related to $\sum_{k=0}^{m} k!\ a_k b_k x^k$ still hold, but s and r exchange their roles. Moreover, in case $f'(0) \neq f(0) = 0$, (c) still holds. Thus Theorem 3 (a) and (b) generalize the second part of [3, Theorem 6, pp. 336-337].

EXAMPLE 2. Let $h(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ and let $f(x) = (1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$ $(2 \leq m \leq n)$. By Theorem 3 (b) the polynomial $g(x) = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\ x^k$ has only real simple roots. In fact, $x^m g(x^{-1}) = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\ x^{m-k}$ = the polynomial in Example 1.

COROLLARY 2. *If* $(a_k)_{k=0}^{\infty}$, $a_0 \neq 0$ *or* $a_1 \neq 0$ *is a multiplier sequence of the second kind, then the polynomial* $g_n(x) = \sum_{k=0}^{n} \frac{1}{(n-k)!} \binom{n}{k} a_k x^k$ *has only real simple roots for every positive integer* n.

PROOF. $h(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ has only negative roots and $(a_k)_{k=0}^{\infty}$ is a multiplier sequence of the second kind, so $f(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ has only real roots and $f(0) \neq 0$ or $f'(0) \neq 0$. By applying Theorem 3 (b) to f and h, we get $\sum_{k=0}^{n} k! \binom{n}{k}\binom{n}{k} a_k x^k = n! \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(n-k)!} a_k x^k$.

University of Haifa,
Mt. Carmel, Haifa, ISRAEL.

REFERENCES

[1] T. Craven and G. Csordas, *On the number of real roots of polynomials*, Pacific Journal of Mathematics, Vol. 102, No. 1 (1982), 15-28.

[2] G. Csordas and J. Williamson, *The zeros of Jensen Polynomials are simple*, Proc. Amer. Math. Soc., Vol. 49 (1975), 263-264.

[3] B. Ja Levin, *Distribution of zeros of entire functions*, Transl., Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, R.I., 1964.

[4] M. Marden, *Geometry of polynomials*, Math. Surveys, No. 3. Amer. Math. Soc., Providence, R.I., 1966.

[5] Summer Institute on Entire Functions and Related Parts of Analysis, University of California, San Diego, 1966. *Lecture notes prepared in connection with the Summer Institute on Entire Functions and Related Parts of Analysis held at University of California, San Diego, La Jolla, California, June 27-July 22, 1966*. Amer. Math. Soc., 1966. Problem 26(H. Wilf), 541-542.