COMMON FIXED POINTS OF MAPPINGS AND SET-VALUED MAPPINGS ON METRIC SPACES

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In the following, as in [1], (X, d) is a complete metric space and $B(X)$ is the set of all nonempty subsets of X. The function $\delta(A, B)$ with A and B in $B(X)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

If A consists of a single point a we write

$$\delta(A, B) = \delta(a, B) = \delta(B, a).$$

It follows easily from the definition that

$$\delta(A, B) = \delta(B, A) \geqslant 0,$$
$$\delta(A, B) \leqslant \delta(A, C) + \delta(C, B)$$

for all A, B and C in $B(X)$.

Now let $\{A_n : n = 1, 2, \cdots\}$ be a sequence of nonempty subsets of X. We say that the sequence $\{A_n\}$ *converges* to the subset A of X if:

(i) each point a in A is the limit of a convergent sequence $\{a_n \in A_n : n = 1, 2, \cdots\}$,

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ.

A is then said to be the *limit* of the sequence $\{A_n\}$.

The following lemma was proved in [1].

LEMMA. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converges to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Now let F be a mapping of a complete metric space (X, d) into $B(X)$. We say that the mapping F is *continuous* at the point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is a *continous mapping* of X into $B(X)$ if F is continous at each point x in X. We say that a point z in X is a *fixed point* of F if z is in Fz.

We now prove the following theorem.

THEOREM 1. *Let F and G be mappings of a complete metric space (X, d) into* *$B(X)$ and let I and J be mappings of X into itself satisfying the inequality*

$$\delta(Fx, Gy) \leqslant c \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\} \qquad (1)$$

for all x, y in X, where $0 \leqslant c < 1$. If the mappings F and I commute and the *mappings G and J commute, if the range of I contains the range of G and the* *range of J contains the range of F and if I (or J) is continuous, then F, G, I* *and J have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the* *unique common fixed point of F, G, I and J.*

PROOF. We let $x = x_0$ be an arbitrary point in X and define the sequence $\{x_n\}$ inductively. Having defined the point x_{2n} we choose a point x_{2n+1} with Jx_{2n+1} in $Z_{2n} = Fx_{2n}$. This can be done since the range of J contains the range of F. Next choose a point x_{2n+2} with Ix_{2n+2} in $Z_{2n+1} = Gx_{2n+1}$, which can be done since the range of I contains the range of G. Then using inequality (1), we have

$$\delta(Fx_{2n}, Gx_{2n+1}) \leqslant c \max\{d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1})\}$$
$$\leqslant c \max\{\delta(Gx_{2n-1}, Fx_{2n}), \delta(Fx_{2n}, Gx_{2n+1})\}$$
$$= cd(Gx_{2n-1}, Fx_{2n}),$$

since Ix_{2n} is in Gx_{2n-1}, Jx_{2n+1} is in Fx_{2n} and $c < 1$.

Similarly

$$\delta(Gx_{2n-1}, Fx_{2n}) = \delta(Fx_{2n}, Gx_{2n-1})$$
$$\leqslant c\delta(Fx_{2n-2}, Gx_{2n-1})$$

and so

$$\delta(Fx_{2n}, Gx_{2n+1}) \leqslant c\delta(Gx_{2n-1}, Fx_{2n})$$
$$\leqslant c^{2n}\delta(Fx_0, Gx_1)$$

for $n = 1, 2, \cdots$.

Now for arbitrary $\epsilon > 0$ choose an integer N such that

$$\frac{c^N}{1-c}\delta(Z_0, Z_1) < \epsilon.$$

Then if z_n is an arbitrary point in z_n for $n = 1, 2, \cdots$, it follows that

$$d(z_m, z_n) \leqslant \delta(Z_m, Z_n) < \epsilon$$

for $m, n > N$. It follows that the sequence $\{z_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X, the point z being independent of the particular choice of each z_n. In particular, the sequences $\{Ix_{2n}\}$

and $\{Jx_{2n+1}\}$ will converge to z and further, the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ will converge to the set $\{z\}$. We will now suppose that the mapping I is continuous. Then noting that every sequence $\{z_n\}$, with z_n in Z_n, converges to z, it follows that the sequence $\{I^2x_{2n}\}$ converges to Iz and the sequence $\{IFx_{2n}\} = \{FIx_{2n}\}$ converges to $\{Iz\}$. Using inequality (1) we have

$$\delta(FIx_{2n}, \ Gx_{2n+1}) \leqslant c \ \max\{d(I^2x_{2n}, \ Jx_{2n+1}), \ \delta(I^2x_{2n}, \ FIx_{2n}),$$
$$\delta(Jx_{2n+1}, \ Gx_{2n+1})\}.$$

Letting n tend to infinity and using the lemma we have
$$d(Iz, z) \leqslant cd(Iz, z)$$
and it follows that $Iz = z$. Further

$$\delta(Fz, \ Gx_{2n+1}) \leqslant c \ \max\{d(Iz, Jx_{2n+1}), \ \delta(Iz, Fz), \ \delta(Jx_{2n+1}, \ Gx_{2n+1})\}.$$

Letting n tend to infinity it follows that
$$\delta(Fz, \ z) \leqslant c\delta(z, Fz)$$
and so $Fz = \{z\}$. This means that z is in the range of F and since the range of J contains the range of F there must exist a point z' in X such that
$$Jz' = z$$
and so
$$GJz' = Gz = JGz' \tag{2}$$
since G and J commute. Thus

$$\begin{aligned}\delta(z, \ Gz') &= \delta(Fz, \ Gz') \\ &\leqslant c \ \max\{d(Iz, \ Jz'), \ \delta(Iz, \ Fz), \ \delta(Jz', \ Gz')\} \\ &= c\delta(z, \ Gz')\end{aligned}$$

and so $Gz' = \{z\}$. On using equations (2), it follows that $Gz = \{Jz\}$. Thus

$$\begin{aligned}\delta(z, \ Gz) &= \delta(Fz, \ Gz) \\ &\leqslant c \ \max\{d(Iz, \ Jz), \ \delta(Iz, \ Fz), \ \delta(Jz, \ Gz)\} \\ &= cd(Iz, \ Jz) \\ &= c\delta(z, \ Gz)\end{aligned}$$

from what we have just proved and so
$$Gz = \{z\} = \{Jz\}.$$

We have therefore proved that z is a common fixed point of F, G, I and J and that $Fz = Gz = \{z\}$. The same result of course holds if J is continuous instead of I.

Now suppose that F, G, I and J have a second common fixed point w. Then

$$\delta(Fw,\ Gw) \leqslant c \max\{d(Iw,\ Jw),\ \delta(Iw,\ Fw),\ \delta(Jw,\ Gw)\}$$
$$= c \max\{\delta(w,\ Fw),\ \delta(w,\ Gw)\}$$
$$\leqslant c\delta(Fw,\ Gw).$$

Since $c < 1$, it follows that

$$\delta(Fw,\ Gw) = 0$$

and so

$$Fw = Gw = \{w\}.$$

Thus on using inequality (1) we have

$$d(z,\ w) = \delta(Fz,\ Gw) \leqslant cd(z,\ w)$$

and so z is the unique common fixed point of F, G, I and J. This completes the proof of the theorem.

The following corollary follows immediately.

COROLLARY. *Let F and G be mappings of a complete metric space (X, d) into* $B(X)$ *satisfying the inequality*

$$\delta(Fx,\ Gy) \leqslant c \max\{d(x,\ y),\ \delta(x,\ Fx),\ \delta(y,\ Gy)\}$$

for all x, y in X, where $0 \leqslant c < 1$. Then F and G have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique common fixed point of F and G.

The result of this corollary was given in [2].

THEOREM 2. *Let F and G be mappings of a complete metric space (X, d) into* $B(X)$ *and let I and J be mappings of X into itself satisfying inequality (1) for all x, y in X, where $0 \leqslant c < 1$. If the mappings F and I commute and the mappings G and J commute, if the range of I contains the range of G and the range of J contains the range of F, if F (or G) is continuous and if*

$$\delta(Fx,\ Fx) \leqslant \delta(x,\ Fx) \tag{3}$$
$$(\text{or } \delta(Gx,\ Gx) \leqslant \delta(x,\ Gx) \tag{3'})$$

for all x, y in X, then F, G, I and J have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique common fixed point of F, G, I and J.

PROOF. Define a sequence $\{x_n\}$ as in the proof of theorem 1 so that the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ again converge to z and the sequences $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to $\{z\}$. Supposing that F is continuous we see that the sequences $\{FJx_{2n+1}\}$ and $\{FIx_{2n}\} = \{IFx_{2n}\}$ converge to Fz. Thus

$$\delta(FJx_{2n+1},\ Gx_{2n+1}) \leqslant c \max\{d(IJx_{2n+1},\ Jx_{2n+1}),\ \delta(IJx_{2n+1},\ FJx_{2n+1}),$$
$$\delta(Jx_{2n+1},\ Gx_{2n+1})\}$$

$$\leqslant c \max\{\delta(IFx_{2n}, Jx_{2n+1}), \delta(IFx_{2n}, FJx_{2n+1}), \delta(Jx_{2n+1}, Gx_{2n+1})\},$$

since Jx_{2n+1} is in Fx_{2n} and so IJx_{2n+1} is in IFx_{2n}. Letting n tend to infinity and using the lemma we have

$$\delta(Fz, z) \leqslant c \max\{\delta(Fz, z), \delta(Fz, Fz), d(z, z)\}$$
$$= c\delta(Fz, z)$$

because of inequality (3). It follows that $Fz = \{z\}$. This means that z is in the range of F and so again there exists a point z' such that $Jz' = z$ and equations (2) are satisfied. Further

$$\delta(FJx_{2n+1}, Gz') \leqslant c \max\{d(IJx_{2n+1}, Jz'), \delta(IJx_{2n+1}, FJx_{2n+1}), \delta(Jz', Gz')\}$$
$$\leqslant c \max\{\delta(IFx_{2n}, z), \delta(IFx_{2n}, FJx_{2n+1}), \delta(z, Gz')\}.$$

Letting n tend to infinity it follows from what we have just proved that

$$\delta(z, Gz') \leqslant c\delta(z, Gz')$$

and so $Gz' = \{z\}$. From equations (2) we now see that $Gz = \{Jz\}$. Further

$$\delta(Fx_{2n}, Gz) \leqslant c \max\{d(Ix_{2n}, Jz), \delta(Ix_{2n}, Fx_{2n}), \delta(Jz, Gz)\}.$$

Letting n tend to infinity it follows from what we have just proved that

$$\delta(z, Gz) \leqslant c\delta(z, Gz)$$

and so $Gz = \{z\} = \{Jz\}$. The point z is therefore in the range of G and since the range of I contains the range of G there exists a point z'' in X such that $Iz'' = z$. Thus

$$\delta(Fz'', z) = \delta(Fz'', Gz)$$
$$\leqslant c \max\{d(Iz'', Jz), \delta(Iz'', Fz''), \delta(Jz, Gz)\}$$
$$= c\delta(z, Fz'')$$

and so $Fz'' = \{z\}$. It follows that

$$FIz'' = Fz = \{z\} = IFz'' = \{Iz\}.$$

We have therefore proved that z is a common fixed point of F, G, I and J.

The same result holds if G is continuous instead of F. That z is the unique common fixed point of F, G, I and J follows as in the proof of theorem 1. This completes the proof of the theorem.

THEOREM 3. *Let S, T, I and J be mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) \leqslant c \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\}$$

for all x, y in X, where $0 \leqslant c < 1$. If the mappings S and I commute and the mappings T and J commute, if the range of I contains the range of T and the range of J contains the range of S and if one of the mappings S, T, I and J is continuous, then S, T, I and J have a unique common fixed point z. Further, z

is the unique common fixed point of S and I and of T and J.

PROOF. Define set-valued mappings F and G on X by
$$Fx = \{Sx\}, \quad Gx = \{Tx\}$$

for all x in X. If I (or J) is continuous then the conditions of theorem 1 are satisfied and it follows that S, T, I and J have a unique common fixed point z.

If F (or G) is continuous then inequality (3) (or inequality (3')) is trivially satisfied and so the conditions of theorem 2 are satisfied. It again follows that S, T, I and J have a unique common fixed point z.

Now suppose that T and J have a second common fixed point w. Then
$$d(z, w) = d(Sz, Tw)$$
$$\leqslant c \max\{d(Iz, Jw), \, d(Iz, Sz), \, d(Iw, Tw)\}$$
$$= cd(z, w).$$

The uniqueness of z follows.

Similarly z is the unique common fixed point of S and I. This completes the proof of the theorem.

For an example to see that z is not necessarily the unique common fixed point of F and I and of G and J in theorems 1 and 2, see the example given in [2] for the particular case when I and J are the identity mappings.

It is possible that the condition that inequality (3) (or (3')) can be omitted from theorem 2, but examples are easily found to show that the other conditions of theorems 1 and 2 cannot be omitted.

We finally prove a theorem for compact metric spaces.

THEOREM 4. *Let F and G be continuous mappings of the compact metric space* (X, d) *into* $B(X)$ *and let I and J be continuous mappings of X into itself satisfying the inequality*

$$\delta(Fx, Gy) < \max\{d(Ix, Jy), \, \delta(Ix, Fx), \, \delta(Jy, Gy)\} \tag{4}$$

for all x, y in X for which the right-hand side of the inequality is positive. If the mappings F and I commute and the mappings G and J commute and if the range of I contains the range of G and the range of J contains the range of F, then F, G, I and J have a unique common fixed point z. Further, $Fz = Gz = \{z\}$ *and z is the unique common fixed point of F, G, I and J,*

PROOF. Suppose first of all that the right-hand side of inequality (4) is positive for all x, y in X. Define the function $f(x, y)$ on $X \times X$ by

$$f(x, y) = \frac{\delta(Fx, Gy)}{\max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}}.$$

Then if $\{(x_n, y_n)\}$ is an arbitrary sequence in $X \times X$ converging to (x, y), it follows easily from the lemma that the sequence $\{f(x_n, y_n)\}$ converges to $f(x, y)$. The function f is therefore a continuous function defined on the compact metric space $X \times X$ and so achieves its maximum value c. Inequality (4) implies that $c < 1$ and it follows that the conditions of theorem 1 are satisfied. Hence F, G, I and J have a unique common fixed point z. Now suppose that the right-hand side of inequality (4) is zero for some x, y in X. Then

$$Fx = Gy = \{Ix\} = \{Jy\}$$

is a singleton $\{z\}$ and so

$$F^2 x = FIx = IFx = \{Iz\}$$

is also a singleton. If $Iz \neq z$ then

$$\begin{aligned} d(Iz, z) &= \delta(F^2 x, Gy) \\ &< \max\{\delta(IFx, Jy), \delta(IFx, F^2 x), \delta(Jy, Gy)\} \\ &= d(Iz, z). \end{aligned}$$

giving a contradiction. It follows that

$$\{Iz\} = \{z\} = Fz$$

and so z is a common fixed point of F and I.

Similarly, we can prove that G and J have a common fixed point z'. If $z \neq z'$ we have

$$\begin{aligned} d(z, z') &= \delta(Fz, Gz') \\ &< \max\{d(Iz, Jz'), \delta(Iz, Fz), \delta(Jz', Gz')\} \\ &= d(z, z'). \end{aligned}$$

giving a contradiction. It follows that $z = z'$ is a common fixed point of F, G, I and J. Finally suppose that F, G, I and J have a second distinct common fixed point w. Then if $Fw \neq \{w\}$ or $Gw \neq \{w\}$

$$\begin{aligned} \delta(Fw, Gw) &< \max\{d(Iw, Jw), \delta(Iw, Fw), \delta(Jw, Gw)\} \\ &\leq \delta(Fw, Gw), \end{aligned}$$

since $Iw = Jw = w$ is in Fw and Gw, giving a contradiction. Thus $Fw = Gw = \{w\}$ and so

$$d(z, w) = \delta(Fz, Gw) < d(z, w),$$

giving a contradiction. The uniqueness of z follows. This completes the proof of the theorem.

The following corollaries follow easily.

COROLLARY 1. *Let F and G be continuous mappings of the compact metric space* (X, d) *into* $B(X)$ *satisfying the inequality*

$$\delta(Fx, Gy) < \max\{d(x, y), \delta(x, Fx), \delta(y, Fy)\}$$

for all x, y *in* X *for which the right-hand side of the inequality is positive. Then F and G have a unique common fixed point* z. *Further,* $Fz = Gz = \{z\}$ *and* z *is the unique common fixed point of F and G.*

COROLLARY 2. *Let S, T, I and J be continuous mappings of a compact metric space* (X, d) *into itself satisfying the inequality*

$$d(Sx, Ty) < \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\}$$

for all x, y *in* X *for which the right-hand side of the inequality is positive. If the mappings S and I commute and the mappings T and J commute and if the range of I contains the range of T and the range of J contains the range of S, then S, T, I and J have a unique common fixed point* z. *Further,* z *is the unique common fixed point of S and I and of T and J.*

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