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QUOTIENT ADDITIVE PROPERTIES IN TOPOLOGICAL SPACES

by Norman Levine

1. Introduction

Suppose that R is an equivalence relation on a space X and $[x]$ and X/R are compact for each x in X. If the projection map $p: X \to X/R$ is closed, then X is compact. See theorem 4.2.

This result is not new. This paper is concerned with the more general question: If R is an equivalence relation on a space X and if P is a topological property, does X have property P when X/R and $[x]$ have property P for each x in X?

We will call P a quotient additive property when the answer is in the affirmative.

The following properties are shown to be quotient additive: indiscreteness, discreteness, T_0, T_1, connectedness, total disconnectedness and singleton path components.

The following properties fail to be quotient additive even when $p: X \to X/R$ is an open map: Lindelof, countable compactness, extremally disconnected and cofinite.

The following properties fail to be quotient additive even when $p: X \to X/\mathrm{R}$ is both open and closed: paracompact, metacompact, Hausdorff, regular and normal.

The following properties fail to be quotient additive even when $p: X \to X/R$ is closed: second axiom, separable, metrizable and path connected.

If $p: X \to X/R$ is closed, the following properties are shown to be quotient additive: Lindelof, countably compactness and (if X is first axiom) sequential compactness.

In section 5, $X \times Y$ is shown to be a fixed point space if X is a T_1 fixed point space and Y is strongly separable (see definition 5.1).

2. General properties

We begin with

THEOREM 2.1. *If X/R and $[x]$ are indiscrete for each x in X, then X is indiscrete.*

PROOF. Suppose that $O \neq O \neq X$ and O is open in X. Then for each x in X, $[x] \subseteq O$ or $[x] \cap O = \phi$ since $[x]$ is indiscrete. Let $O^* = \{[x] : [x] \subseteq O\}$. Then $O \neq O^* \neq X/R$ and O^* is open in X/R, a contradiction.

THEOREM 2.2. *Let X/R be discrete and let $[x]$ be discrete for each x in X. Then X is discrete.*

PROOF. $\{x\}$ is open in $[x]$ and $[x]$ is open in X since $\{[x]\}$ is open in X/R. Thus $\{x\}$ is open in X.

THEOREM 2.3. *Let X/R be a T_0-space and let $[x]$ be a T_0-space for each x in X. Then X is a T_0-space.*

PROOF. Let $x \neq y$. Case 1. $[x] \neq [y]$. Then we can assume that there exists an open set O^* in X/R such that $[x] \in O^*$ and $[y] \notin O^*$. Then $x \in p^{-1}[O^*]$ and $y \notin p^{-1}[O^*]$. Case 2. $[x] = [y]$. Then there exists an open set O in X such that $x \in O \cap [x]$ and $y \notin O \cap [x]$. Then $x \in O$ and $y \notin O$.

THEOREM 2.4. *Let X/R and $[x]$ be T_1-spaces for each x in X. Then X is a T_1-space.*

PROOF. Modify theorem 2.2.

We note that neither T_2 nor T_3 nor T_4 nor T_5 may be substituted for T_1 in theorem 2.4 (see example 3.9).

THEOREM 2.5. *Let X/R and $[x]$ be connected for each x in X. Then X is connected.*

PROOF. Suppose that $X = O \cup V$ where O and V are nonempty disjoint sets in X. Then $x \in O$ implies that $[x] \subseteq O$ and $y \in V$ implies that $[y] \subseteq V$. Let $O^* = \{[x] : x \in O\}$ and $V^* = \{[y] : y \in V\}$. Then $X/R = O^* \cup V^*$ and O^* and V^* are disjoint nonempty open sets in X/R, a contradiction.

THEOREM 2.6. *Let X/R and $[x]$ be totally disconnected for each x in X. Then X is totally disconnected.*

PROOF. Let $A \subseteq X$ and suppose that A has more than one point. Case 1. $A \subseteq [x]$ for some x in X. Then A is disconnected since $[x]$ is totally disconnected. Case 2. $A \subseteq [x]$ for no x in X. Then $p[A]$ contains more than one point in X/R and hence is disconnected. It follows then that A is disconnected.

THEOREM 2.7. *Let X/R and $[x]$ have singleton path components for each x in X. Then X has singleton path components.*

PROOF. Let f : $[0, 1] \to X$ be continuous. We will show that f is a constant. Now $p \circ f : [0, 1] \to X/R$ is a constant and hence $f[[0, 1]] \subseteq [x]$ for some x in X. Since $[x]$ has singleton path components, f is a constant.

3. Some examples

The next two lemmas will be useful for counterexample purposes.

LEMMA 3.1. *Let $W = X \times Y$ and let $W/R = \{A_x : x \in X\}$ where $A_x = \{x\} \times Y$. Then W/R is homeomorphic to X and A_x is homeomorphic to Y for each x in X.*

PROOF. Let $p : W \to X$ via $p((x, y)) = x$ and let $q : W \to W/R$ via $q((x, y)) = A_x$. Let $h : W/R \to X$ via $h(A_x) = x$. Then h is clearly bijective and $h \circ q = p$. Since q and p are quotient maps, h is a homeomorphism.

That A_x is homeomorphic to Y is well known.

LEMMA 3.2. *Let $X = I \times I$ Where $I = [0, 1]$ and let X have the dictionary order topolcgy. Let $X/R = \{A_x : x \in I\}$ where $A_x = \{x\} \times I$. Then A_x is homeomorphic to I for each x in X and X/R is homeomorphic to I.*

PROOF. Let $p : X \to I$ by $p(x, y) = x$. p is clearly continuous and onto, and since X is compact and I is hausdorff, $p : X \to I$ is a closed map and hence a quotient map. Let $q : X \to X/R$ by $q(x, y) = A_x$. Since X/R is given the quotient topology, $q : X \to X/R$ is a quotient map. Let $h : X/R \to I$ by $h(A_x) = x$. As in lemma 3.1, h is a homeomorphism.

DEFINITION 3.3. A property p will be termed *finitely nonproductive* if the product of two spaces with property p need not have property p.

THEOREM 3.4. *Let p be a finitely nonproductive propery. If X/R and $[x]$ have property p for each x in X, then X need not have property p even if $p : X \to X/R$ is open.*

PROOF. Use lemma 3.1.

COROLLARY 3.5. *Let p be any one of the following properties: Lindelof, count-*

ably compact, paracompact, extremally disconnected, normal or cofinite. If X/R and $[x]$ have property p for each x in X, then X need not have property p even if $p: X \to X/R$ is open.

PROOF. All of the above properties are finitely nonproductive.

COROLLARY 3.6. *Let X, X/R and $[x]$ be as in lemma 3.2. Then X/R and $[x]$ are separable, second axiom, metrizable and path connected. X has none of these properties and $p: X \to X/R$ is closed.*

EXAMPLE 3.7. Let $X = \{x : 0 \leqslant x \leqslant 1$ or $1 < x < 2$ and x is rational$\}$. Let $Y = [0. 1]$ and let $f: X \to Y$ as follows: $f(x) = x$ if $0 \leqslant x \leqslant 1$ $f(x) = x - 1$ if $1 < x < 2$. Let X have the usual topology and let Y have the quotient topology. For each y in Y, $f^{-1}[y]$ is a singleton set or a doubleton set and hence is compact, locally compact, locally connected and sequentially compact. Y also has all of these properties. X has none of these properties.

EXAMPIE 3.8. Let $X = \{a, 1, 2, \cdots, n, \cdots\}$ and let a subset of X be open iff it is empty or contains a. Let R be the equivalence relation whose equivalence classes are $[a] = \{a\}$ and $[1] = \{1, 2, \cdots, n, \cdots]$. Now $[a]$ and $[1]$ are paracompact and metacompact. X/R is a two point space and hence is metacompact and every open cover has a locally finite open refinement. But $\{\{a, x\} : x \text{ in } X\}$ is an open cover of X with no refinement which is locally finite or point finite. Note that $p: X/R$ is both open and closed.

EXAMPLE 3.9. Let $X = \{a, b, 1, 2, 3, \cdots, n, \cdots\}$ and let a subject A of X be open iff $A \cap \{a, b\} = \phi$ or $A \cap \{a, b\} \neq \phi$ and $\mathscr{C}A$ is finite, $\mathscr{C}$ denoting the complement operator. Let $[a] = \{a, b\}$ and $[n] = \{n\}$ for $n = 1, 2, \cdots$. Then X/R is homeomorphic to $\left\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\right\}$ with the usual topology and $p: X \to X/R$ is both open and closed and X/R and $[x]$ are T_2, T_3, T_4 and T_5. X has none of these properties.

4. $p: X \to X/R$ closed

When $p: X \to X/R$ is a closed map, certain covering properties are well behaved.

We begin with a well known property of closed maps.

LEMMA 4.1. *Let $f: X \to Y$ be a closed map and let $f^{-1}[y] \subseteq O$, O* being an open set in X. Then there exists an open set U in Y such that $y \in U$ and $f^{-1}[U] \subseteq O$.

PROOF. Let $U=\mathscr{C}f[\mathscr{C}O]$.

THEOREM 4.2. *Let $p: X \to X/R$ be closed and let X/R and $[x]$ be compact for each x in X. Then X is compact.*

PROOF. See [2], theorem 5.3, page 236 or modify the proof of the next theorem.

THEOREM 4.3. *If $p: X \to X/R$ is a closed map and if X/R and $[x]$ are Lindelof for each x in X, then X is Lindelof.*

PROOF. Let $X=\{O_\alpha : \alpha \in \Delta\}$ where O_α is open in X. Since $[x]$ is a Lindelof space, there exists a countable set $\Delta([x]) \subseteq \Delta$ such that $p^{-1}[[x]]=[x] \subseteq \cup \{O_\alpha : \alpha \in \Delta([x])\}$. By lemma 4.2, there exists an open set $U([x])$ in X/R such that $[x] \in U([x])$ and $p^{-1}[U[x]] \subseteq \cup \{O_\alpha : \alpha \in \Delta([x])\}$. Since X/R is Lindelof, $X/R = \cup \{U([x]_i) : i \geqslant 1\}$ and thus $X = \cup \{O_\alpha : \alpha \in \cup \{\Delta([x]_i) : i \geqslant 1\}\}$.

THEOREM 4.4. *If $p: X \to X/R$ is a closed map and if X/R and $[x]$ are countably compact for each x in X, then X is countably compact.*

PROOF. Let $\{E_n : n \geqslant 1\}$ be a sequence of closed sets with the finite intersection property. We may assume that $E_n \supseteq E_{n+1}$ for all n. Now $\{p[E_n] : n \geqslant 1\}$ is a sequence of closed sets in X/R with the finite intersection property. Since X/R is countably compact, there exists a $[x]$ in X/R such that $[x] \in \cap \{p[E_n] : n \geqslant 1\}$. Then $\{[x] \cap E_n : n \geqslant 1\}$ is a sequence of sets closed in $[x]$ with the finite intersection property and since $[x]$ is countably compact, $\cap \{[x] \cap E_n : n \geqslant 1\} \neq \phi$. Thus $\cap \{E_n : n \geqslant 1\} \neq \phi$ and X is countably compact.

COROLLARY 4.5. *If $p: X \to X/R$ is closed and if X/R and $[x]$ are sequentially compact for each x in X and if X is a first axiom space, then X is a first axiom space.*

PROOF. X/R and $[x]$ are countably compact for each x in X. Then X is countably compact by theorem 4.4 and hence X is sequentially compact since first axiom is assumed.

5. A fixed point theorem

We first note that $X \times Y$ need not be a fixed point space when X and Y are fixed point spaces. See [1].

Using some of the earlier ideas, we will give a sufficient condition for

$X \times Y$ to be a fixed point space.

DEFINITION 5.1. A space X will be called *strongly separable* iff there exists a point x^* in X such that $X=c(\{x^*\})$ and $x \neq x^*$ implies that $\{x\}$ is closed. c denotes the closure operator.

EXAMPLE 5.2. Let X be a set and $x^* \in X$. Let $\mathscr{T}_1=\{0 \subseteq X : 0=O \text{ or } x^* \in 0\}$. Let $\mathscr{T}_2=\{0 \subseteq X : 0=O \text{ or } x^* \in 0 \text{ and } \mathscr{C}0 \text{ is finite}\}$. Let $\mathscr{T}_3$ be any topology on X for which $\mathscr{T}_2 \subseteq \mathscr{T}_3 \subseteq \mathscr{T}_1$. Then $(X, \mathscr{T}_3)$ is strongly separable.

LEMMA 5.3. *Let X be a strongly separable space with distinguished point x^*. Then X is a fixed point space.*

PROOF. Let $f : X \to X$ be continuous. If $f(x^*)=x^*$, we are done. So assume that $f(x^*)=y \neq x^*$. Then $f(y) \in f(c(\{x^*\})) \subseteq c\, f[\{x^*\}]=c(\{y\})=\{y\}$. Thus $f(y)=y$.

LEMMA 5.4. *Let X be a space and $\{A_\alpha : \alpha \in \Delta\}$ a partition of X. Suppose further that for each $\alpha \in \Delta$, there is a point x_α in X such that $A_\alpha=c(\{x_\alpha\})$. If $f : X \to Y$ is continuous and $\alpha \in \Delta$, there exists a $\beta \in \Delta$ such that $f[A_\alpha] \subseteq A_\beta$.*

PROOF. Let $f(x_\alpha) \in A_\beta$. Then $f[A_\alpha]=f[c(\{x_\alpha\})] \subseteq c\, f[\{x_\alpha\}] \subseteq c\, A_\beta = A_\beta$.

THEOREM 5.5. *Let X be a T_1-fixed point space and let Y be a strongly separable space with distinguished point y^*. Then $X \times Y$ is a fixed point space.*

PROOF. Let R be the equivalence relation induced by the partition $\{\{x\} \times Y : x \in X\}$. Note that $\{x\} \times Y=c(\{(x, y^*)\})$. Now let $f : X \times Y \to X \times Y$ be continuous and let $f/R : (X \times Y)/R \to (X \times Y)/R$ be the map induced by lemma 5.4. f/R is continuous. By lemma 3.1, $(X \times Y)/R$ is homeomorphic to X and hence is a fixed point space. There exists then an x in X such that $f/R(\{x\} \times Y)=\{x\} \times Y$. Then $f | \{x\} \times Y : \{x\} \times Y \to \{x\} \times Y$ and since $\{x\} \times Y$ is a fixed point space, $f(x, y)=(x, y)$ for some y in Y.

The Ohio state University
Columbus, Ohio 43210
U.S.A.

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