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ON HP-MINIMAL AND HP-CLOSED SPACES

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1. Introduction

For a given topological property G and a set X, the set $G(X)$ of all topologies on X which have property G is partially ordered by set inclusion. A space $(X, \mathscr{T})$ is *minimal* G(*maximal* G) if $\mathscr{T}$ is a minimal (maximal) element in $G(X)$. A topology $\mathscr{T}'$ on the set X is *finer* than a topology $\mathscr{T}$ if $\mathscr{T}' \supset \mathscr{T}$, and $\mathscr{T}$ is said to be *coarser* than $\mathscr{T}'$. Parhomenko [8] introduced the notion of minimal spaces when he showed that compact Hausdorff spaces are minimal Hausdorff. This result, together with that of Hewitt [5] that compact Hausdorff spaces are maximal compact, has led to a great deal of topological activity. The reader is referred to the surveys of Berri, Porter and Stephenson [2] and Larson and Andima [6]. The term *P-space* was introduced as an abbreviation for pseudodiscrete space by Gillman and Henriksen [4] to mean a completely regular Hausdorff space X on which every continuous real-valued function is constant on some neighbourhood of each point of X. Their interest in P-spaces was from the point of view of the ring $C(X)$ of continuous real-valued functions on X. In a P-space the intersection of countably many open sets is again an open set. We take this condition to be the definition of such spaces. Thus by a *P-space* we mean a topological space in which every G_δ set is open. Misra [7] has considered some of the topological properties of such spaces, although he required T_1 separation as part of his definition and we do not. Misra's P-spaces coincide with quasi-P-spaces of Cameron [3], who uses the term P-space to mean a completely regular T_1 topological space in which every G_δ set is open. Our attention was drawn to P-spaces when we noticed that a Lindelof Hausdorff P-space is maximal Lindelof and minimal Hausdorff P. This result is given by both Cameron [3, Theorem 7.6] and Misra [7, Proposition 4.2(f)]. In a sense this paper can be regarded as a continuation

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of the last four sections of Cameron's paper. The notations and definitions we use are standard with the convention that no separation axioms will be assumed unless explicitly stated. The closure of a set A with respect to the topology $\mathscr{T}$ is denoted $\mathscr{T}$ cl A, the interior of A with respect to $\mathscr{T}$ is $\mathscr{T}$ int A, the complement of A in X is $X-A$, and the relative topology of the set A with respect to the topology $\mathscr{T}$ is $\mathscr{T}/A$. In section 2 we provide some basic results for these classes of spaces. Section 3 gives co-space characterizations, and section 4 presents two miscellaneous results. In [9], we considered projective and injective objects in the category of *HP*-closed spaces.

2. Hausdorff p-closed and minimal hausdorff p-spaces

Cameron [3] has considered many properties of these classes of spaces. In this section we introduce our terms and notations and prove a few new results. Henceforth we shall denote a Hausdorff P-space as an *HP-space*. If the *HP*-space $(X, \mathscr{T})$ is closed in any imbedding of X into an *HP*-space we say that X is *HP-closed.* Cameron uses the term L-closed for this notion, but we prefer to retain the close analogy with H-closed spaces. If $\mathscr{C}$ is a family of subsets of a topological space $(X, \mathscr{T})$, then $\mathscr{C}$ is called an *almost cover* of X if the collection of closures of members of $\mathscr{C}$ covers X. $\mathscr{C}$ is said to have an *almost subcover* $\mathscr{C}'$ of X if $\mathscr{C}'\subset\mathscr{C}$ and $\mathscr{C}'$ is an almost cover in its own right.

As we have already observed the following result is due to Cameron [3, Theorem 7.6].

THEOREM 2.1. *A Lindelof HP-space is maximal Lindelof and minimal HP.*

Cameron has shown that a maximal Lindelof space need not be Hausdorff [3, Example 7.3] but is a P-space [3, Theorem 7.4], that a minimal *HP*-space need not be Lindelof [3, Example 9.1], and that a Lindelof *HP*-space need not be minimal Hausdorff nor minimal P [3, Examples 7.4 and 7.5]. The following example shows that the classes of H-closed and *HP*-closed spaces are distinct.

EXAMPLE 2.2. Let $X=[1, \omega]$ where ω is the first countably infinite ordinal. Let $\mathscr{T}$ be the discrete topology on X, and let $(X, \mathscr{T}')$ be the one-point compactification of the discrete subspace $[1, \omega]$. Then $(X, \mathscr{T}')$ is compact

Hausdorff and hence H-closed. But it is not HP-closed for it is not a P-space. On the other hand, $(X, \mathscr{T})$ is a Lindelof HP-space and hence HP-closed (see Theorem 2.3 (iii) below). However, $(X, \mathscr{T})$ is not H-closed, for this together with the regularity of $(X, \mathscr{T})$ would imply compactness of $(X, \mathscr{T})$.

We say that a filter (base) is a *σ-filter (base)* if the filter (generated by the filterbase) is closed under countable intersections. The following result is given by Cameron [3, page 244(c)].

THEOREM 2.3. *Let $(X, \mathscr{T})$ be an HP-space. The following are equivalent.*

(i) $(X, \mathscr{T})$ is HP-closed.

(ii) Every open σ-filterbase on $(X, \mathscr{T})$ has an adherent point.

(iii) Every open cover of $(X, \mathscr{T})$ has a countable almost subcover.

This result can be used to show that the continuous image of an HP-closed space is HP-closed, and also that if $(X, \mathscr{T})$ is HP-closed, and $G \in \mathscr{T}$ then $\mathscr{T}\mathrm{cl}G$ is HP-closed.

THEOREM 2.4. *If $(X, \mathscr{T})$ is HP-closed there is a toplogy $\mathscr{T}'$ on X coarser than $\mathscr{T}$ such that $(X, \mathscr{T}')$ is minimal HP. Furthermore, if $(X, \mathscr{T})$ is Urysohn then $(X, \mathscr{T}')$ is regular.*

PROOF. Let $\mathscr{B} = \{X - \mathscr{T}\mathrm{cl}G : G \in \mathscr{T}\}$. Then $\mathscr{B}$ is a base for a topology $\mathscr{T}'$ on X, such that $(X, \mathscr{T}')$ is an HP-space. Furthermore, $(X, \mathscr{T}')$ is HP-closed since the indentity $i : (X, \mathscr{T}) \to (X, \mathscr{T}')$ is continuous. It is easily verified that $(X, \mathscr{T}')$ is minimal-HP.

Now suppose $(X, \mathscr{T})$ is Urysohn, and let $x \in A \in \mathscr{T}'$. There is a $G \in \mathscr{T}$ such that $x \in X - \mathscr{T}\mathrm{cl}G \subset A$. For each point $y \in \mathscr{T}\mathrm{cl}G$ there are $\mathscr{T}$ open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $(\mathscr{T}\mathrm{cl}U_y) \cap (\mathscr{T}\mathrm{cl}V_y) = \phi$. Thus $\{V_y \cap \mathscr{T}\mathrm{cl}G : y \in \mathscr{T}\mathrm{cl}G\}$ is an open cover of $\mathscr{T}\mathrm{cl}G$ which is HP-closed. So there is a countable set of points in $\mathscr{T}\mathrm{cl}G$, $\{y_i : i \in N\}$, such that $\mathscr{T}\mathrm{cl}G \subset \mathscr{T}\mathrm{cl}[\cup\{V_{y_i} : i \in N\}]$. Now let $F = \cap\{U_{y_i} : i \in N\}$. Then we have that $x \in F \subset X - \cup\{\mathscr{T}\mathrm{cl}V_{y_i} : i \in N\} \subset X - \mathscr{T}\mathrm{cl}G \subset A$ and F is $\mathscr{T}'$ closed. Thus $(X, \mathscr{T}')$ is regular.

REMARKS 2.5. The topology $\mathscr{T}'$ of the previous theorem is called the *semi-regularization* of $\mathscr{T}$ and usually denoted by $\mathscr{T}_s$. If $(X, \mathscr{T})$ is a topological space such that $\mathscr{T} = \mathscr{T}_s$, it is called *semi-regular*. We observe that if $(X, \mathscr{T})$ is a P-space, so if $(X, \mathscr{T}_s)$.

The first three conditions in the next result have been shown to be equivalent by Cameron [3, Theorem 9.2]. The proof of the equivalence of the first and the fourth conditions is straightforward.

THEOREM 2.6. *For an HP-space* $(X, \mathscr{T})$ *the following are equivalent.*

(*i*) $(X, \mathscr{T})$ *is minimal-HP.*

(*ii*) $(X, \mathscr{T})$ *is HP-closed and semi-regular.*

(*iii*) *Every open* σ*-filter base on* $(X, \mathscr{T})$ *with a unique adherent point converges.*

(*iv*) *Every one-to-one continuous map from* $(X, \mathscr{T})$ *to an HP-space is an embedding.*

COROLLARY 2.7. *If a minimal HP-space is Urysohn then it is Lindelof.*

PROOF. Follows from Theorems 2.6, 2.4 and 2.3 (iii) and the observation of Cameron [3, p. 244(d)].

We now describe an example we shall need in the sequel.

EXAMPLE 2.8. Let $A=[2_a, \Omega_a)$, $B_1=[2_b, \Omega_b)$, $C_1=[2_c, \Omega_c)$, $B=B_1\times B_1$, $C=C_1\times C_1$, and $X=\{0, 1\}\cup A\cup B\cup C$. Let x_a, x_b, x_c denote three points having the same ordinal x with $x_a\in A$, $x_b\in B_1$, $x_c\in C_1$, and let P be a countable subset of B_1 and Q a countable subset of C_1. We now define subsets of X as follows:

$$B(P, x_b)=\{x_b\}\times(B_1-P)$$
$$C(Q, x_c)=\{x_c\}\times(C_1-Q)$$
$$V(P, Q, x_a)=\{x_a\}\cup B(P, x_b)\cup C(Q, x_c)$$
$$V(P, 0)=\{0\}\cup[(B_1-P)\times B_1]$$
$$V(Q, 1)=\{1\}\cup[(C_1-Q)\times C_1]$$

We now define a topology $\mathscr{T}$ on X as follows:

(i) If $p\in B\cup C$ then $\{p\}\in\mathscr{T}$.

(ii) The $\mathscr{T}$ open neighbourhoods of x_a, 0 and 1 are the families $\{V(P, Q, x_a)\}$, $\{V(P, 0)\}$ and $\{V(Q, 1)\}$ respectively where P and Q are arbitrary countable subsets of B_1 and C_1 respectively.

$(X, \mathscr{T})$ is an *HP*-space. To show that it is minimal *HP*, we show that if is *HP*-closed and semi-regular. To prove that $(X, \mathscr{T})$ is *HP*-closed we use Theorem 2.3 (iii). If $\mathscr{C}$ is an open cover of $(X, \mathscr{T})$, there are sets V_0 and V_1 in $\mathscr{C}$ such that $0\in V_0=V(P_0, 0)$ and $1\in V_1=V(Q_1, 1)$. Let $K_0=\{x_a : x_b\in P_0\}$, $K_1=\{x_a : x_c\in Q_1\}$ and $K=K_0\cup K_1$. We form three countable subcollections from $\mathscr{C}$. Let $\mathscr{C}_1=\{V(P_0, Q_1, x_a) : x_a\in K\}$, $\mathscr{C}_2=\{\{x_b\}\times P_0 : x_a\in K\}$ and $\mathscr{C}_3=\{\{x_c\}\times$

$Q : x_a \in K\}$. If $Z = V_0 \cup V_1 \cup \{V : V \in \mathscr{C}_i,\ i=1, 2, 3\}$ then $Z = (X-A) \cup K = X-(A-K)$. Moreover, if $x \in A-K$ then $x \in \mathscr{T}\mathrm{cl}Z$, so that $\mathscr{C}$ has a countable almost subcover and $(X, \mathscr{T})$ is *HP*-closed. To show that $(X, \mathscr{T})$ is semiregular, consider some neighbourhood $V(P_0, 0)$ of 0. Let $M = \{x_a \in A : x_b \in B_1 - P_0\}$. Then $V(P_0, 0) \cap V(P, Q, x_a) \neq \phi$ for $x_a \in M$. Thus $M \subset \mathscr{T}\mathrm{cl}V(P_0, 0)$. Indeed, $\mathscr{T}\mathrm{cl}V(P_0, 0) = M \cup V(P_0, 0)$. Furthermore, $\mathscr{T}\mathrm{int}(\mathscr{T}\mathrm{cl}V(P_0, 0)) = V(P_0, 0)$ as each point of M is a boundary point of $V(P_0, 0)$. Thus every $\mathscr{T}$ neighbourhood of 0 (and similarly of 1) is regular-open. If $p \in B \cup C$ then $\mathscr{T}\mathrm{cl}\ \{p\} = \{p\} \in \mathscr{T}$. Clearly, $\mathscr{T}\mathrm{cl}V(P, Q, x) = V(P, Q, x)$. Hence $(X, \mathscr{T})$ is semi-regular.

3. Co-space characterizations

The notion of co-topology was intrcduced by de Groot]1], and applied to the study of minimal topological spaces by Strecker and Viglino [10, 11, 12]. We restrict our discussion here to *P*-spaces and their co-spaces. Our definition of cospace is slightly different from the definition given by de Groot. Let $\mathscr{B}$ be a base for the topological space $(X, \mathscr{T})$. The coarsest *P*-space topology on X generated by all sets of the form $\{X - \mathscr{T}\mathrm{cl}U : U \in \mathscr{B}\}$ is called the *co-topology of* $\mathscr{T}$ *with respect to* $\mathscr{B}$, and is denoted $\mathscr{T}(\mathscr{B})$. So by definition, our co-topologies are *P*-spaces. If a co-topology $\mathscr{T}(\mathscr{B})$ has the topological property G then $(X, \mathscr{T})$ is said to be *co-G*. If every co-space of $(X, \mathscr{T})$ satisfies the property G, then $(X, \mathscr{T})$ is called *totally co-G*.

THEOREM 3.1. *If* $(X, \mathscr{T}(\mathscr{B}))$ *is Lindelof, then every* σ*-filter base consisting of element of* $\mathscr{B}$ *has an adherent point.*

PROOF. Suppose $\mathscr{T}(\mathscr{B})$ is Lindelof and that $\mathscr{F}$ is a σ-filter base such that $\mathscr{F} \subset \mathscr{B}$ and $\mathrm{ad}(\mathscr{F}) = \phi$. Then $\{X - \mathscr{T}\mathrm{cl}F : F \in \mathscr{F}\}$ is a $\mathscr{T}(\mathscr{B})$ open cover of X, so it has a countable subcover $\{X - \mathscr{T}\mathrm{cl}F_i : i \in N\}$. Thus $\cup\{X - \mathscr{T}\mathrm{cl}F_i : i \in N\} = X$ implies $\cap\{\mathscr{T}\mathrm{cl}F_i : i \in N\} = \phi$ so that $\cap\{F_i : i \in N\} = \phi$, which contradicts the fact that $\mathscr{F}$ is a σ-filter base. Hence $\mathrm{ad}(\mathscr{F}) \neq \phi$, as required.

COROLLARY 3.2. *Every totally co-Lindelof HP-space is HP-closed.*

THEOREM 3.3. *The following are equivalent for an HP-space* $(X, \mathscr{T})$.

(*i*) $(X, \mathscr{T})$ *is HP-closed.*

(*ii*) $(X, \mathscr{T})$ *is totally co-G for any property G such that minimal-HP*$\Rightarrow G \Rightarrow$*HP*

(*iii*) *All the co-spaces of* $(X, \mathscr{T})$ *are homeomorphic.*

PROOF. (i) implies (ii). Suppose $(X, \mathscr{T})$ is *HP*-closed. Let there be a non-Hausdorff co-topology $\mathscr{T}(\mathscr{B})$ (which is a *P*-space, by definition). Clearly there exist points p, $q \in X$ such that every $\mathscr{T}(\mathscr{B})$-neighbourhood of p meets every $\mathscr{T}(\mathscr{B})$-neighbourhood of q. Now for $x \in X-\{p, q\}$ there are sets U_x, $U_{1,x}$, $U_{2,x} \in \mathscr{B}$ such that $x \in U_x$, $p \in U_{1,x} \cap (X-\text{cl}(U_x \cup U_{2,x}))$ and $q \in U_{2,x} \cap (X-\text{cl}(U_{1,x} \cup U_x))$. Write $W_x=(X-\text{cl}(U_{2,x} \cup U_x)) \cap (X-\text{cl}(U_{1,x} \cup U_x))$. Now $\{W_x | x\ X-\{p, q\}\}$ is a non-empty collection of nonempty open sets with the property that every countable subfamily has a non-empty intersection so that this family generates an open σ-filterbase $\mathscr{F}_1$ with empty adherence. (Here if X is countable then $(X, \mathscr{T})$ is discrete so that $(X, \mathscr{T}(\mathscr{B}))$ is also discrete for any base and in this case also, the space is totally co-*HP*). Thus if $(X, \mathscr{T})$ is *HP*-closed, then it is totally co-*HP*. Also the co-topology $\mathscr{T}(\mathscr{T})=\mathscr{T}_s$ is miaimal-*HP* by Theorem 2.4. If $\mathscr{B}$ is a base for $\mathscr{T}$, then $\mathscr{B} \subset \mathscr{T}$ so that $\mathscr{T}(\mathscr{B}) \subset \mathscr{T}(\mathscr{T})$. Since $\mathscr{B}(\mathscr{B})$ is *HP*, $\mathscr{T}(\mathscr{B})=\mathscr{T}(\mathscr{T})$ for all co-topologies $\mathscr{T}(\mathscr{B})$. Thus $(X, \mathscr{T})$ is totally co-minimal-*HP*.

(ii) implies (i). If $(X, \mathscr{T})$ is not *HP*-closed there is an open σ-filterbase $\mathscr{F}$ on $(X, \mathscr{T})$ with empty adherence. Thus $\{X-\mathscr{T}\text{cl}F : \in \mathscr{F}\}$ is an open cover of X. Then $\mathscr{B}=\{W \cap (X-\mathscr{T}\text{cl}F) : F \in \mathscr{F},\ W \in \mathscr{T}\}$ is a base for $\mathscr{T}$, so that $\mathscr{T}(\mathscr{B})$ is *HP*. Hence for distinct points x and y in X there are disjoint $\mathscr{T}(\mathscr{B})$ open sets V_x and V_y containing x and y respectively. So $V_x=X-\mathscr{T}\text{cl}V_1$ and $V_y=X-\mathscr{T}\text{cl}V_2$, where $V_1=W_1 \cap (X-\mathscr{T}\text{cl}F_1)$ and $V_2=W_2 \cap (X-\mathscr{T}\text{cl}F_2)$. Since $V_x \cap V_y=\phi$ we have that $X=\mathscr{T}\text{cl}V_1 \cap \mathscr{T}\text{cl}V_2 \subset \mathscr{T}\text{cl}(X-\mathscr{T}\text{cl}F_1) \cap \mathscr{T}\text{cl}(X-\mathscr{T}\text{cl}F_2) \subset (X-\mathscr{T}\text{int}F_1) \cup (X-\mathscr{T}\text{int}F_2)$ as $\mathscr{T}\text{cl}(X-\mathscr{T}\text{cl}F) \subset \mathscr{T}\text{cl}(X-F)=X-\mathscr{T}\text{int}F$. Thus $X=(X-F_1) \cup (X-F_2)$ since F_1 and F_2 are $\mathscr{T}$ open. Hence $F_1 \cap F_2=\phi$ which contradicts being a filterbase.

(i) implies (iii). The above argument shows that if $(X, \mathscr{T})$ is *HP*-closed, $\mathscr{T}(\mathscr{B})=\mathscr{T}(\mathscr{T})$ so that all co-spaces are homeormophic.

(iii) implies (i). If all co-spaces are homeomorphic all the co-spaces are *HP*, so that the space is totally co-*HP* and hence *HP*-closed.

THEOREM 3.4. *The following are equivalent for an HP space* $(X, \mathscr{T})$

(i) $(X, \mathscr{T})$ *is HP-closed and Urysohn.*

(ii) $(X, \mathscr{T})$ *is HP and every σ-filter base generated by sets of the form* $\{\mathscr{T}\text{cl}G : G \in \mathscr{T}\}$ *has an adherent point.*

(iii) $(X, \mathscr{T})$ *is Urysohn P and all the co-spaces of* $(X, \mathscr{T})$ *are homeomorphic*

(iv) $(X, \mathscr{T})$ *is totally co-G for any property G such that Lindelof* $HP \Rightarrow G \Rightarrow$

Urysohn P.

(v) $(X, \mathscr{T})$ *is completely Hausdorff and HP-closed.*

(vi) $(X, \mathscr{T})$ *is HP-closed and for each pair of disjoint HP-closed subsets M and N in* $(X, \mathscr{T})$ *there is a continuous function* $f: (X, \mathscr{T}) \to [0, 1]$ *such that* $f(M)=1$ *and* $f(N)=0$.

PROOF. (i) and (iii) are equivalent by the previous theorem.

(i) implies (iv). By Theorem 2.4 the co-topology $\mathscr{T}(\mathscr{T})$ is minimal-*HP* and regular, and therfore Urysohn. Thus, by Corollary 2.7, $\mathscr{T}(\mathscr{T})$ is Lindelof. Hence, by (iii), all co-topologies are Lindelof *HP*.

(iv) implies (i). $(X, \mathscr{T})$ is totally co-Urysohn *P*, so, in particular $\mathscr{T}(\mathscr{T})$ is Urysohn, which implies that $\mathscr{T}$ is Urysohn. If $(X, \mathscr{T})$ is totally co-Urysohn *P* it is totally co-*HP*, and hence by Theorem 3.3 it is *HP*-closed.

(i) implies (ii). As before, $\mathscr{T}(\mathscr{T})$ is Lindelof *HP*, so that (ii) holds by Theorem 3.1.

(ii) implies (i) is clear from Theorem 2.3.

(i) implies (vi). If A is *HP*-closed in $(X, \mathscr{T})$ then it is closed in $(X, \mathscr{T}(\mathscr{T}))$. As before $\mathscr{T}(\mathscr{T})$ is Lindelof *HP* and therefore T_4. Thus M and N are disjoint closed subsets of $(X, \mathscr{T}(\mathscr{T}))$, so that normality gives disjoint $\mathscr{T}(\mathscr{T})$ open sets U and V with $M \subset U$ and $N \subset V$. Since $\mathscr{T}(\mathscr{T}) \subset \mathscr{T}$, $U, V \in \mathscr{T}$ so that a Urysohn Lemma type argument gives the result.

That (vi) implies (v), and (v) implies (i) are clear.

THEOREM 3.5. *The following are equivalent for an HP-space* $(X, \mathscr{T})$.

(i) $(X, \mathscr{T})$ *is minimal-HP.*

(ii) $(X, \mathscr{T})$ *is HP-closed and* $\mathscr{T}=\mathscr{T}(\mathscr{B})$ *for some base* $\mathscr{B}$ *of* $\mathscr{T}$.

(iii) $(X, \mathscr{T})$ *is totally co-minimal-HP and semi-regular.*

(iv) $\mathscr{T}=\mathscr{T}(\mathscr{B})$ *for every co-topology* $\mathscr{T}(\mathscr{B})$.

PROOF. (i) implies (ii). If $(X, \mathscr{T})$ is minimal-*HP*, then Theorem 2.6 implies it is *HP*-closed and semi-regular. Thus $\mathscr{T}=\mathscr{T}_s=\mathscr{T}(\mathscr{T})$.

(ii) implies (i). If $\mathscr{T}=\mathscr{T}(\mathscr{B})$ for some base $\mathscr{B}$ of $\mathscr{T}$, then $\mathscr{T}(\mathscr{B}) \subset \mathscr{T}(\mathscr{T})$ implies $\mathscr{T}=\mathscr{T}(\mathscr{T})=\mathscr{T}_s$, so that $(X, \mathscr{T})$ is semi-regular. But $(X, \mathscr{T})$ is *HP*-closed so by Theorem 2.6 it is minimal-*HP*.

(i) and (iii) are equivalent by Theorem 3.3 (i) and (iv) and Theorem 2.6.

(i) implies (iv). If $(X, \mathscr{T})$ is minimal-*HP* it is *HP*-closed, and hence by Theorem 3.3, it is totally co-*HP*. Thus $\mathscr{T}(\mathscr{B})$ is *HP* for every co-topology

$\mathscr{T}(\mathscr{B})$ of $\mathscr{T}$. But $\mathscr{T}(\mathscr{B})\subset\mathscr{T}$ contradicts the minimal-*HP* property of $(X, \mathscr{T})$ unless $\mathscr{T}(\mathscr{B})=\mathscr{T}$.

(iv) implies (i). If $\mathscr{T}=\mathscr{T}(\mathscr{B})$ for every co-topology, then $(X, \mathscr{T})$ is totally co-*HP* and hence *HP*-closed by Theorem 3.3. Morever, $\mathscr{T}=\mathscr{T}(\mathscr{T})=\mathscr{T}_s$ so that $(X, \mathscr{T})$ is semi-regular, and thus it is minimal-*HP* by Theorem 2.6.

Now, we give an example of a non-Lindelof space which is *HP*-closed and Urysohn.

EXAMPLE 3.6. Consider the space X of Example 2.8. Let $Y=\{0\}\cup A\cup B$. If $\mathscr{T}$ is as defined in Example 2.8, then $(Y, \mathscr{T}(Y))$ is a Urysohn *HP*-closed space which is not Lindelof.

4. Two miscellaneous results

THEOREM 4.1. *If $(X, \mathscr{T})$ is minimal T_4P then X is $\mathscr{T}$-Lindelof.*

PROOF. Suppose the space is not Lindelof. Consider a σ-filter base $\mathscr{F}$ of closed sets with empty intersection. Let x be some fixed point in X. Define $\mathscr{T}'$ on X as follows. $U\in\mathscr{T}'$ if $x\notin U\in\mathscr{T}$ or $U\in\mathscr{T}'$ if $x\in U\in\mathscr{T}$ and $F\subset U$ for some $F\in\mathscr{F}$. Clearly $\mathscr{T}'$ defines an *HP*-space on X which is normal and $\mathscr{T}'\subset\mathscr{T}$.

The notion of a one-point Lindelof extension of locally Lindelof *HP* spaces was discussed by Cameron in [3]. This notion is used to prove the following theorem.

THEOREM 4.2. *In the class of all HP topologies on X, a minimal locally Lindelof HP topology is Lindelof.*

PROOF. Let $X^+=X\cup\{p\}$, where $(X, \mathscr{T})$ is a minimal locally Lindelof *HP*-space, be the one-point Lindelof extension as given by Cameron in [3], supposing X is not Lindelof. Let $\mathscr{G}$ be the filter base of open neighbourhoods of p in $(X^+, \mathscr{T}^+)$. Let $\mathscr{G}'=\{G\cap X|G\in\mathscr{G}\}$. Let a be a fixed point in X. Define $\mathscr{T}^*$ on X such that $U\in\mathscr{T}^*$, if $a\notin U\in\mathscr{T}$ or $U\in\mathscr{T}^*$, if $U=V\cup W$, $a\in W\in\mathscr{T}$ and $V\in\mathscr{G}'$. Clearly $\mathscr{T}^*$ is a Lindelof *HP* topology on X such that $\mathscr{T}^*$ is a proper subset of $\mathscr{T}$.

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REFERENCES

[1] J.M. Aarts and J. de Groot, et al., *Colloquium co-topologies*, Mathematisch Centrum (Amsterdam) syllabus zwA, 1964.

[2] M.P. Berri, J.R. Porter and R.M. Stephenson, Jr., *A survey of minimal topological spaces*, General topology and its relation to modern analysis and algebra III, Proc. Conference Topology (Kanput, 1968), Academic Press, New York, 1970, 93—114.

[3] D.E. Cameron, *Maximal and minimal topologies*, Trans. Amer. Math. Soc. 160(1971), 229—248.

[4] L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. 77(1954), 340—362.

[5] E. Hewitt, *A problem of set theoretic topology*, Duke Math. J. 10(1943), 309—333.

[6] R.E. Larson and S.J. Andima, *The lattice of topologies: a survey*, Rocky Mountain J. Math. 5(1675), 177—198.

[7] A.K. Misra, *A topological view of P-spaces*, Gen. Top. Appl. 2(1972), 349—362.

[8] A.S. Parhomenko, *Uber eineindeutige stetige Abbildungen*, Mat. Sb. 5(47) (1939), 197—210, (Russian).

[9] T.G. Raghavan and I.L. Reilly, *On projective and injective objects in some topological categories*, Lecture Notes in Mathematics 719, Springer-Verlag (1979), 308—315.

[10] G.E. Strecker, *Co-topologies and generalized compactness conditions*, Ph.D. dissertation, Tulane University, 1966.

[11] G.E. Strecker and G. Viglino, *Co-topology and minimal Hausdorff spaces*, Proc. Amer. Math. Soc., 21(1969), 569—574.

[12] G. Viglino, *A co-topological application to minimal spaces* Pacific J. Math., 27(1968), 197—200.