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A NEW INTRINSIC CHARACTERIZATION OF HYPERCYLINDERS IN EUCLIDEAN SPACES

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1. Introduction

Let R and Q denote respectively the *Riemann-Christoffel curvature tensor* and the *Ricci tensor* of a Riemannian manifold M. Various authors then studied the condition $R\cdot Q=0$, where R acts as a derivation on Q: for hypersurfaces, see P. Ryan [5] [6] [7], S. Tanno [10] and Y. Matsuyama [2] [3]; for conformally flat spaces, see K. Sekigwa & H. Takagi [8] and R.L. Bishop & S.I. Goldberg [1] ; for Bochner-Kaehler spaces, see H. Takagi & Y. Watanabe [9]. In this article we study the curvature condition $Q\cdot R=0$ for hypersurfaces of Euclidean spaces, and prove the following.

THEOREM. *A hypersurface of a Euclidean space is a hypercylinder if and only if it satisfies* $Q\cdot R=0$.

2. Proof of the theorem

Let M be a hypersurface of an $(n+1)$-dimensional Euclidean space. Let ξ be a local normal section on M. In the following X, Y, Z denote vector fields which are tangent to M. Then the formulas of Gauss and Weingarten are given by

(1) $$D_X Y=\nabla_X Y+h(X, Y)\xi$$

and

(2) $$D_X\xi=-AX,$$

where D is the Euclidean connection on E^{n+1} and ∇ is the Levi Civita connection on M. The second fundamental tensor A is related to the second fundamental form h by $h(X, Y)=g(AX, Y)$, where g is the Riemannian metric on M. Let $X\wedge Y$ denote the endomorphism $Z\to g(Z, Y)X-g(Z, X)Y$. Then the curvature tensor R of M is given by the equation of Gauss:

(3) $$R(X, Y)=AX\wedge AY.$$

Since A is symmetric there exists an orthonormal frame $e_1, e_2, \cdots, e_n$ consisting of eigenvectors, i.e. such that

$$Ae_i = \lambda_i e_i, \tag{4}$$

$(k, j, i, t \in \{1, 2, \cdots, n\})$, whereby $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the principal curvatures of M. The hypersurface M is called a *hypercylinder* if at least $n-1$ of its principal curvatures are zero. By (4) the equation of Gauss implies that

$$R(e_i, e_j) = c_{ij} e_i \wedge e_j, \quad c_{ij} = \lambda_i \lambda_j, \tag{5}$$

(see also [4]). Consequently

$$R(e_i, e_j) e_k = c_{ij}(\delta_{jk} e_i - \delta_{ik} e_j), \tag{6}$$

such that the Ricci tensor Q of M is determind by

$$Qe_i = \mu_i e_i, \quad \mu_i = \lambda_i (\sum_t \lambda_t - \lambda_i). \tag{7}$$

The condition

$$Q \cdot R = 0 \tag{8}$$

means that

$$(Q \cdot R)(e_i, e_j) e_k = 0, \tag{9}$$

or equivalently,

$$Q(R(e_i, e_j) e_k) - R(Qe_i, e_j) e_k - R(e_i, Qe_j) e_k - R(e_i, e_j) Qe_k = 0. \tag{10}$$

The theorem follows at once from the following lemmas.

LEMMA 1. $Q \cdot R = 0 \Longleftrightarrow \forall i \neq j : \mu_j c_{ij} = 0$.

PROOF. From (6), (7) and (10) it follows that (8) is equivalent to

$$c_{ij}\{(\mu_i + \mu_k)\delta_{ik} e_j - (\mu_j + \mu_k)\delta_{jk} e_i\} = 0, \tag{11}$$

which clearly implies the statement of Lemma 1.

LEMMA 2. $Q \cdot R = 0 \Longleftrightarrow n-1$ *of the principal curvatures are* 0.

PROOF. The implication $\Leftarrow$ follows immediately from Lemma 1. (We remark that this is trivial since every hypercylinder of a Euclidean space is locally flat). To show the converse, we use (5) and (7) in Lemma 1 to obtain

$$\forall i \neq j : \lambda_i \lambda_j (\sum_t \lambda_t - \lambda_j) = 0 \tag{12}$$

as an equivalent condition for $Q \cdot R = 0$. Consequently also

$$\forall i \neq j : \lambda_j \lambda_i (\sum_t \lambda_t - \lambda_i) = 0. \tag{13}$$

From (12) and (13) we find that

$$\forall i \neq j : \lambda_i \lambda_j (\lambda_i - \lambda_j) = 0. \tag{14}$$

A. Suppose that, for some λ, we have

$$\forall i : \lambda_i = \lambda. \tag{15}$$

Then from (12) it follows that

$$(n-1)\lambda^3=0, \tag{16}$$

which shows that λ must be zero. (In this case M is a *totally geodesic hypersurface*, i.e. a linear subspace of E^{n+1}).

B. Suppose that precisely $n-1$ of the principal curvatures are zero. Then clearly (12) is satisfied. (In this case M is a *proper hypercylinder* in E^{n+1}).

C. Suppose that

$$\lambda_1=\lambda_2=\cdots=\lambda_p=0 \tag{17}$$

for some p,

$$0\leqslant p<n-1, \tag{18}$$

and assume that all other principal curvatures $\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_n$ are non-zero. From (14) it then follows that all the latter curvatures must be equal, say

$$\lambda_{p+1}=\lambda_{p+2}=\cdots=\lambda_n=\lambda, \quad (\lambda\neq 0). \tag{19}$$

Having considered the cases $p=n$ and $p=n-1$ in A and B, at present we have at least two principal curvatures, say λ_{n-1} and λ_n, which are equal to λ. For these curvatures, (12) means that

$$(n-p-1)\lambda^3=0, \tag{20}$$

which by (18) yields a contradiction. Therefore only the cases A and B can occur.

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