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On Factor States on a Fixed Point Algebra of a UHF Algebra by the Torus Action II

by

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>*Abstract*<

A study is made of a special type of C^*-dynamical systems, consisting of a class n^∞ uniformly hyperfinite C^*-algebra $\mathscr{A}$, the torus group $G=T^d$ $(1\leqq d\leqq n-1)$ and a natural product action of G on $\mathscr{A}$ by $*$-automorphisms. We give some conditions for product states on the fixed point algebra $\mathscr{A}^G$ of $\mathscr{A}$ by G to be factorial.

1. Introduction

We study a special type of C^*-dynamical system $(\mathscr{A}, G, \alpha)$ where $\mathscr{A}$ is a uniformly hyperfinite (UHF) n^∞ C^*-algebra and G is the group of d dimensional torus, $1\leqq d\leqq n-1$ and α is a natural product action of G on $\mathscr{A}$ by $*$-automorphisms. We consider a product state ω on $\mathscr{A}$ and the fixed point algebra $\mathscr{A}^G$ of $\mathscr{A}$ under the action α. We determine the structure of $\mathscr{A}^G$ (Theorem 3.1) and of the relative commutant of $\mathscr{A}_m{}^G$ in $\mathscr{A}^G$ (Theorem 3.9), as developed in (7).

A product state ω on $\mathscr{A}$ is a factor state (9). We would like to give conditions for the restriction ω^G of ω to $\mathscr{A}^G$ to be factorial. For the case of $n=2$, Baker and Powers have given a necessary and sufficient condition for this (1, 3). For the case of a general integer n with $d=1$, the author gave the condition for this (7). In this paper we give a necessary condition (Theorem 4.1) and a sufficient condition (Theorem 5.2) for ω^G to be factorial for general n and $1\leqq d\leqq n-1$.

2. Definitions and notations

Let B_m; $m=1, 2, \ldots$ be type I_n factors $*$-isomorphic to the $n\times n$ complex matrix algebra B_0. Let $\mathscr{A}$ be the infinite tensor product C^*-algebra $\bigotimes_{m=1}^{\infty} B_m$ that is the type n^∞

uniformly hyperfinite (UHF) C^*-algebra (8). Each B_m is identified as a subalgebra of $\mathscr{A}$. Let φ_m be the *-isomorphism of B_0 onto B_m. Let $\{e_{ij};\ i,j=1,2,\dots,n\}$ be a matrix unit of B_0. Then $\{\varphi_m(e_{ij});\ 1\leqq i,j\leqq n\}$ is a matrix unit of B_m for $m\geqq 1$. Remark that for $m\neq t$ $\varphi_m(e_{ij})$ commutes with $\varphi_t(e_{kl})$ and $\{\varphi_m(e_{ij});\ m\geqq 1,\ 1\leqq i,\ j\leqq n\}$ generates $\mathscr{A}$. Put $\mathscr{A}_m=\bigotimes_{k=1}^{m} B_k$. Then $\mathscr{A}_m$ is identified with the C^*-subalgebra of $\mathscr{A}$ and $\mathscr{A}$ is a C^*-closure of $\bigcup\{\mathscr{A}_m;\ m\geqq 1\}$.

Let Λ_m be the family of mappings of $\{1,2,\dots,m\}$ into $\{1,2,\dots,n\}$. Put $K_m=\{k=(k_1,\dots,k_d)\in\mathbf{Z}^d;\ k_j\geqq 0,\ j=1,2,\dots,d,\ \Sigma k_j\leqq m\}$ for $m\geqq 1$. Put $\Lambda_{m,k}=\{\phi\in\Lambda_m;\ \#\{i;\ \phi(i)=j\}=k_j,\ j=1,2,\dots,d\}$ for $k\in K_m$ where # denotes the cardinal number of a set. Then Λ_m is a disjoint union of $\Lambda_{m,k}$, $k\in K_m$. For $\phi,\ \psi\in\Lambda_m$ put $f_{\phi\psi}=\varphi_1(e_{\phi(1)\psi(1)})\cdots\varphi_m(e_{\phi(m)\psi(m)})$. Then $\{f_{\phi\psi};\ \phi,\ \psi\in\Lambda_m\}$ is a matrix unit of $\mathscr{A}_m$.

Let G be the group of d dimensional torus T^d, where $1\leqq d\leqq n-1$. We define the action α of G on $\mathscr{A}$ as *-automorphisms. For $t=(t_1,\dots,t_d)\in G=[0,\ 2\pi)^d$ the diagonal unitary matrix U_t in B_0 is defined as (j,j) component is $\exp(it_j)$ for $1\leqq j\leqq d$ and as the other diagonals are 1. Let $\alpha_t^{(m)}$ be the adjoint *-automrphism of B_m defined by $\alpha_t^{(m)}(x)=\varphi_n(U_t)x\varphi_n(U_t)^*$, $x\in B_m$. Then we can define the *-automorphism $\bigotimes_m \alpha_t^{(m)}$ of $\mathscr{A}$ denoted by α_t. We now constructed a C^*-dynamical system $(\mathscr{A},\ G,\ \alpha)$ (See(6)). We define the fixed point algebra $\mathscr{A}^G$ of $\mathscr{A}$ under the action α by $\mathscr{A}^G=\{x\in\mathscr{A};\ \alpha_t(x)=x$ for all $t\in G\}$. Then $\mathscr{A}^G$ is an approximately finite dimensional (AF) C^*-algebra and is the C^*-closure of $\bigcup\{\mathscr{A}_m{}^G;\ m\geqq 1\}$ where $\mathscr{A}_m{}^G=(\mathscr{A}_m)^G$. We will determine the structure of $\mathscr{A}^G$ by the description of the Bratteli diagram (4).

Let ω_m be a state of B_m for $m\geqq 1$. Then we can define $\otimes\omega_m$ the infinite tensor product state of $\mathscr{A}$ denoted by ω. ω_m is the restriction of ω to B_m and satisfies that $\omega(x_1\otimes\cdots\otimes x_m)=\omega(x_1)\cdots\omega(x_m)$ for $x_j\in B_j$, $1\leqq j\leqq m$. We denote ω^G the restriction of ω to $\mathscr{A}^G$. Then ω^G is a state of $\mathscr{A}^G$.

Let $\mathscr{B},\mathscr{C}$ be C^*-subalgebras of $\mathscr{A}$. Then we denote $\mathscr{B}\vee\mathscr{C}$ the C^*-algebra generated by $\mathscr{B}$ and $\mathscr{C}$. $\mathscr{B}^c$ denotes the relative commutant of $\mathscr{B}$ in $\mathscr{A}$, *i.e.*, $\mathscr{B}^c=\{x\in\mathscr{A};\ xy=yx$ for all $y\in\mathscr{B}\}$. These notations is sometimes used for $\mathscr{B},\mathscr{C}$ only subsets of $\mathscr{A}$.

Let ω be a state of a C^*-algebra $\mathscr{A}$. Then we can canonically construct the cyclic *-representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ with a cyclic vector f_0 such that $\omega(x)=(\pi(x)f_0,\ f_0)$ for $x\in\mathscr{A}$. We say that ω is factorial if the von Neumann algebra $\pi(\mathscr{A})''$ generated by $\pi(\mathscr{A})$ is a factor.

3. Structure of $\mathscr{A}^G$

The fixed point algebra $\mathscr{A}^G$ of $\mathscr{A}$ under the action α is an AF algebra with the Bratteli diagram given by the following theorem.

Theorem 3.1. The fixed point algebra $\mathscr{A}^G$ is an AF algebra $\overline{\bigcup_{m=1}^{\infty}\mathscr{A}_m{}^G}$. $\mathscr{A}_m{}^G$ has a factor decomposition $\oplus\{M_{m,k};\ k\in K_m\}$ where $M_{m,k}$ is a factor of type I_t, $t=\#\Lambda_{m,k}$. The multiplicity p with which $M_{m,k}$ is partially embedded into $M_{m+1,l}$ is given by for $k\in K_m$, $l\in K_{m+1}$

$$p=\begin{cases} n-d & \text{if } l=k \\ 1 & \text{if } l \text{ is of the form; } l_b=k_b+1 \text{ for a single } 1\leqq b\leqq d \\ & \qquad\text{and } l_c=k_c \text{ for all } 1\leqq c\leqq d,\ c\neq b \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\{\varphi_l(e_{ij});\ i,j=1,2,\dots,n\}$ be the matrix unit of B_l; $l\geqq 1$. By definition, $\{f_{\phi\psi}=\varphi_1(e_{\phi(1)\psi(1)})\cdots\varphi_m(e_{\phi(m)\psi(m)});\ \phi,\psi\in\Lambda_m\}$ forms a matrix unit of $\mathscr{A}_m$. Now we have for $t\in G$

$$\alpha_t(f_{\phi\psi})=\exp(i\sum_{l=1}^{d} t_l\ (\#\{j;\ \phi(j)=l\}-\#\{j;\ \psi(j)=l\}))f_{\phi\psi}.$$

Therefore for ϕ, $\psi\in\Lambda_{m,k}$ we have $f_{\phi\psi}\in\mathscr{A}_m{}^G$. Let $M_{m,k}$ be the linear span generated by $\{f_{\phi\psi};\ \phi,\psi\in\Lambda_{m,k}\}$. Then $\mathscr{A}_m{}^G$ is a direct sum of $M_{m,k}$; $k\in K_m$ as vector spaces. Now we have that

$$f_{\phi\psi}f_{\phi'\psi'}=\delta_{kl}\delta_{\psi\phi'}f_{\phi\psi'},\quad f_{\phi\psi}{}^*=f_{\psi\phi}\ \text{ for } \phi,\psi\in\Lambda_{m,k},\ \ \phi',\psi'\in\Lambda_{m,l}$$

and $\Sigma\{f_{\phi\phi};\ \phi\in\Lambda_{m,k}\}$ is a multiplicative unit of $M_{m,k}$. Hence $\{f_{\phi\psi};\ \phi,\psi\in\Lambda_{m,k}\}$ is a matrix unit for $M_{m,k}$ and $M_{m,k}$ is a factor of type I_t; $t=\#\Lambda_{m,k}$. Since $\varphi_{m+1}(e_{11}+\dots+e_{nn})=1$, we have for $\phi,\psi\in\Lambda_{m,k}$

$$f_{\phi\psi}=\sum_{b=1}^{d} f_{\phi\psi}\varphi_{m+1}(e_{bb})+\sum_{b=d+1}^{m} f_{\phi\psi}\varphi_{m+1}(e_{bb}).$$

Define ϕ_b, $\psi_b\in\Lambda_{m+1}$ such that

$$\phi_b(q)=\begin{cases}\phi(q) & \text{for } q=1,2,\dots,m \\ b & \text{for } q=m+1\end{cases} \qquad \text{and } \psi_b \text{ in the same way.}$$

Then we have that for $d+1\leqq b\leqq n$, $\phi_b\in\Lambda_{m+1,k}$ holds and for $1\leqq b\leqq d$, $\phi_b\in\Lambda_{m+1,l}$ holds where l is of the form such that $l_b=k_b+1$ for a single $1\leqq b\leqq d$ and $l_c=k_c$ for all $1\leqq$

$c\neq b\leqq d$. The equality $f_{\phi\psi}=\sum_{b=1}^{d}f_{\phi_b\psi_b}+\sum_{b=d+1}^{n}f_{\phi_b\psi_b}$ shows that the multiplicity with which $M_{m,k}$ is partially embedded into $M_{m+1,l}$ is given as in the statement of the theorem. The proof is complete. Q.E.D.

Put $N(m,b)=\sum_{j=1}^{m}\varphi_j(e_{bb})$ for $1\leqq b\leqq d$. Then $N(m,b)$ is an element of $\mathscr{A}_m^G$. We get a characterization of elements of $\mathscr{A}_m^G$.

Lemma 3.2. $x\in\mathscr{A}_m$ commutes with $N(m,b)$ for all $1\leqq b\leqq d$ if and only if x is in $\mathscr{A}_m^G$.

Proof. $x\in\mathscr{A}_m$ can be written as $\sum c_{\phi\psi}f_{\phi\psi}$ where $c_{\phi\psi}$'s are complex numbers and $\phi,\psi\in\Lambda_m$. Then we have for $1\leqq b\leqq d$

$$\begin{aligned}xN(m,b)-N(m,b)x&=\sum c_{\phi\psi}\sum_{j=1}^{m}(f_{\phi\psi}\varphi_j(e_{bb})-\varphi_j(e_{bb})f_{\phi\psi})\\&=\sum c_{\phi\psi}\sum_j(\delta_{b\psi(j)}\varphi_j(e_{\phi(j)b})-\delta_{b\phi(j)}\varphi_j(e_{b\psi(j)}))\prod_{k\neq j}\varphi_k(e_{\phi(k)\psi(k)})\\&=\sum c_{\phi\psi}(\#\{j;\ \psi(j)=b\}-\#\{j;\ \phi(j)=b\})f_{\phi\psi}.\end{aligned}$$

Hence x commutes with $N(m,b)$ if and only if $c_{\phi\psi}=0$ holds for all $\phi,\psi\in\Lambda_m$ with $\#\{j;\ \phi(j)=b\}\neq\#\{j;\ \psi(j)=b\}$. Consequently x commutes with $N(m,b)$ for all $1\leqq b\leqq d$ if and only if $x\in\mathscr{A}_m^G$ by Theorem 3.1. Q.E.D.

Directly we have by Lemma 3.2,

Lemma 3.3. The center $Z(\mathscr{A}_m^G)$ of $\mathscr{A}_m^G$ is the C^*-algebra generated by $N(m,b)$, $1\leqq b\leqq d$.

Lemma 3.4. $\mathscr{A}_m^G=\mathscr{A}_m\cap\{N(m,b);\ 1\leqq b\leqq d\}^c$ holds.

We shall characterize the relative commutant of $\mathscr{A}_m^G$ in $\mathscr{A}_w$. As in the proof of (10) Lemma 3.2, the relative commutant $(\mathscr{A}_m)^c$of $\mathscr{A}_m$ in $\mathscr{A}$ is given by

Lemma 3.5. $(\mathscr{A}_m)^c=\bigotimes_{i=m+1}^{\infty}B_i$ holds.

Lemma 3.6. $(\mathscr{A}_m^G)^c\cap\mathscr{A}_w=\bigotimes_{i=m+1}^{w}B_i\vee\{N(m,b);\ 1\leqq b\leqq d\}$ holds for integers $m<w$.

Proof. We know that $\mathscr{A}_m^G=\mathscr{A}_m\cap\{N(m,b);\ 1\leqq b\leqq d\}^c$. Since our situation is finite dimensional, we obtain by Lemma 3.5.

$$(\mathscr{A}_m^G)^c\cap\mathscr{A}_w=((\bigotimes_{i=m+1}^{\infty}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}^{cc})\cap(\bigotimes_{i=1}^{w}B_i)=\bigotimes_{i=m+1}^{w}B_i\vee\{N(m,b);\ 1\leqq b\leqq d\}.$$

Lemma 3.7. $(\mathscr{A}_m^G)^c=(\bigotimes_{i=m+1}^{\infty}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}$ holds.

Proof. By Lemma 3.2, we know $N(m,b)\in(\mathscr{A}_m^G)^c$ and $(\mathscr{A}_m)^c\subset(\mathscr{A}_m^G)^c$. Hence we obtain $(\bigotimes_{i=m+1}^{\infty}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}\subset(\mathscr{A}_m^G)^c$ by Lemma 3.5.

Conversely we choose $x\in(\mathscr{A}_m^G)^c$. Then we can choose a sequence $x_N\in\mathscr{A}_N$ such that $x_N\longrightarrow x$ as $N\longrightarrow\infty$. Hence for all $y\in\mathscr{A}_m^G$ we have $x_Ny-yx_N=(x_N-x)y+y(x-x_N)\longrightarrow 0$ as $N\longrightarrow\infty$. We will construct x_N'; $N\geqq m+1$ such that $x_N'\longrightarrow x$ and $x_N'\in(\bigotimes_{i=m+1}^{\infty}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}$. Put $x_N'=\sum_{k\in K_m}\sum_{\phi\in\Lambda_{m,k}}f_{\phi\psi}x_Nf_{\psi\phi}$ where ψ's are fixed for each $k\in K_m$ in the summation. Then we have $x_N'\in\mathscr{A}_N$ and $x_N'f_{\eta\phi}=f_{\eta\psi}x_Nf_{\psi\phi}=f_{\eta\phi}x_N'$ for all $\eta,\phi\in\Lambda_{m,k}$. Hence we conclude $x_N'\in(\mathscr{A}_m^G)^c\cap\mathscr{A}_N=(\bigotimes_{i=m+1}^{N}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}\subset(\bigotimes_{i=m+1}^{\infty}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}$ by Lemma 3.6. On the other hand it holds that

$$\begin{aligned}\|x_N-x_N'\|&=\|\sum_k\sum_\phi f_{\phi\phi}x_N-\sum_k\sum_\phi f_{\phi\psi}x_Nf_{\psi\phi}\|\leqq\sum_k\sum_\phi\|f_{\phi\psi}(f_{\psi\phi}x_N-x_Nf_{\psi\phi})\|\\&\leqq\sum_k\sum_\phi(\|f_{\psi\phi}(x_N-x)\|+\|(x-x_N)f_{\psi\phi}\|)\leqq 2\#K_m\cdot Max\{\#\Lambda_{m,k};k\in K_m\}\|x-x_N\|\end{aligned}$$

and so

$$\|x-x_N'\|\leqq\|x-x_N\|+\|x_N-x_N'\|\leqq(1+2\#K_m\cdot Max\{\#\Lambda_{m,k};\ k\in K_m\})\|x-x_N\|\longrightarrow 0$$

as $N\longrightarrow\infty$. The proof is complete. Q.E.D.

Lemma 3.8. It holds for integers $m<w$ that

$$(\mathscr{A}_m^G)^c\cap\mathscr{A}_w^G=(\bigotimes_{i=m+1}^{w}B_i)^G\vee Z(\mathscr{A}_w^G)=(\bigotimes_{i=m+1}^{w}B_i)^G\vee\{N(m,b);\ 1\leqq b\leqq d\}.$$

Proof. By Lemma 3.3, the second equality is clear. Hence it suffices to show that $(\mathscr{A}_m^G)^c\cap\mathscr{A}_w^G\subset(\bigotimes_{i=m+1}^{w}B_i)^G\vee Z(\mathscr{A}_w^G)$ since the reverse inclusion is trivial.

Choose $x\in(\mathscr{A}_m^G)^c\cap\mathscr{A}_w^G\subset(\mathscr{A}_m^G)^c\cap\mathscr{A}_w=(\bigotimes_{i=m+1}^{w}B_i)\vee\{N(m,b);\ 1\leqq b\leqq d\}$ by Lemma 3.6.

We can write as a finite sum $x=\sum y_kz_k$, $y_k\in\bigotimes_{i=m+1}^{w}B_i$, $z_k\in Z(\mathscr{A}_w^G)$. Define the projection of norm 1 β of $\mathscr{A}_w$ onto $\mathscr{A}_w^G$ such that $\beta=\int_G\alpha_t\,dt$ where dt is the usual normalized Lebesgue measure on G. Since $x\in\mathscr{A}_w^G$, we have $\beta(x)=x$. But we can write as $\beta(x)=\beta(\sum y_kz_k)=\sum\beta(y_k)z_k$. Because $y_k\in\bigotimes_{i=m+1}^{w}B_i$, we have $\beta(y_k)\in(\bigotimes_{i=m+1}^{w}B_i)^G$.

As $x=\sum\beta(y_k)z_k$, we conclude $x\in(\bigotimes_{i=m+1}^{\infty} B_i)^G \vee Z(\mathscr{A}_m{}^G)$. Q.E.D.

Now we can give a formulation of the relative commutant of $\mathscr{A}_m{}^G$ in $\mathscr{A}^G$.

Theorem 3.9. It holds that

$$(\mathscr{A}_m{}^G)^c \cap \mathscr{A}^G = (\bigotimes_{i=m+1}^{\infty} B_i)^G \vee Z(\mathscr{A}_m{}^G) = (\bigotimes_{i=m+1}^{\infty} B_i)^G \vee \{N(m,b);\ 1\leqq b\leqq d\}.$$

Proof. The proof of $(\mathscr{A}_m{}^G)^c \cap \mathscr{A}^G \subset (\bigotimes_{i=m+1}^{\infty} B_i)^G \vee Z(\mathscr{A}_m{}^G)$ is the same as that of Lemma 3.7. Conversely we have $Z(\mathscr{A}_m{}^G) \subset (\mathscr{A}_m{}^G)^c \cap \mathscr{A}^G$ and by Lemma 3.5, $(\bigotimes_{i=m+1}^{\infty} B_i)^G \subset \mathscr{A}^G \cap (\bigotimes_{i=m+1}^{\infty} B_i) = \mathscr{A}^G \cap (\mathscr{A}_m)^c \subset \mathscr{A}^G \cap (\mathscr{A}_m{}^G)^c$. Hence the C^*-algebra is generated by $(\bigotimes_{i=m+1}^{\infty} B_i)^G$ and $Z(\mathscr{A}_m{}^G)$ is contained in $(\mathscr{A}_m{}^G)^c \cap \mathscr{A}^G$. Q.E.D.

4. Necessary condition for ω^G to be factorial

In this section we show that if the state ω^G on $\mathscr{A}^G$ is factorial, then $\sum_{j=1}^{\infty}(1-\omega(\varphi_j(A_b))^2 =0$ or ∞ holds for all $1\leqq b\leqq d$ where A_b is a diagonal matrix whose (b,b) component is -1 and other diagonals are 1.

Theorem 4.1. Let ω be a product state of $\mathscr{A}=\otimes B_i$ where B_i; $i\geqq 1$ are $*$-isomorphic to the $n\times n$ complex matrix algebra B_0. Let ω^G be a state of $\mathscr{A}^G$ which is the restriction of ω to the fixed point algebra $\mathscr{A}^G$ of $\mathscr{A}$ under the action of G defined in section 2. If ω^G is factorial, then ω must satisfy $\sum_{j=1}^{\infty}(1-\omega(\varphi_j(A_b))^2)=0$ or ∞ for all $1\leqq b\leqq d$.

Proof. When we assume that for some $1\leqq b\leqq d, 0<\sum_j(1-\omega(\varphi_j(A_b))^2<\infty$, it suffices to show that ω^G is not factorial. Let $(\pi, \mathscr{H}, f_0)$ be the cyclic $*$-representation of $\mathscr{A}^G$ constructed by ω^G. We denote A instead of A_b in our assumption. Put $J_m=\sum_{j=1}^{m}(\varphi_j(A)-\omega(\varphi_j(A)))$ and $U_m(t)=\exp(it\ \pi(J_m))$ for $t\in R$. We shall prove that the unitary $U_m(t)$ converges strongly to a nonscalar unitary operator in $\pi(\mathscr{A}^G)'\cap\pi(\mathscr{A}^G)''$ as $m\longrightarrow\infty$ for some real t.

Fix $x\in\mathscr{A}_r{}^G$ and $f=\pi(x)f_0\in\mathscr{H}$. Let $p>m>r$ be integers. Then

$$\|U_p(t)f-U_m(t)f\|=\|U_m(t)^*U_p(t)f-f\|=\|\exp\ (it\ \pi(\sum_{j=m+1}^{p}(\varphi_j(A)-\omega(\varphi_j(A)))))f-f\|$$
$$=\|i\textstyle\int_0^t U_m(s)^*U_p(s)\pi(L)f\ ds\|\leqq|t|\,\|\pi(L)f\|=|t|\,\|\pi(Lx)f_0\|$$

where $L=\sum_{j=m+1}^{p}(\varphi_j(A)-\omega(\varphi_j(A)))$. Hence we obtain

$$\|U_p(t)f-U_m(t)f\|^2\leqq t^2\omega(x^*L^2x)=t^2\omega(L^2x^*x)=t^2\omega(L^2)\omega(x^*x)$$

since $L\in\bigotimes_{j=m+1}^{p}B_j, x\in\bigotimes_{j=1}^{r}B_j$ and ω is a product state. Now we have

$$L^2=\sum_{j=m+1}^{p}(\varphi_j(A)^2-2\omega(\varphi_j(A))\varphi_j(A)+\omega(\varphi_j(A))^2)+$$
$$\sum_{j\neq h}(\varphi_j(A)\varphi_h(A)-\omega(\varphi_j(A))\varphi_h(A)-\varphi_j(A)\omega(\varphi_h(A))+\omega(\varphi_j(A))\omega(\varphi_h(A)))$$

and $\omega(L^2)=\sum_{j=m+1}^{p}(\omega(\varphi_j(1))-2\omega(\varphi_j(A))^2+\omega(\varphi_j(A))^2)=\sum_{j=m+1}^{p}(1-\omega(\varphi_j(A))^2)$ using the property that ω is a product state. Hence we have $\omega(L^2)\longrightarrow 0$ as $m\longrightarrow\infty$ since $\Sigma(1-\omega(\varphi_j(A))^2)$ converges. We conclude that $\{U_m(t)f; m\geqq 1\}$ is a Cauchy sequence in $\mathscr{H}$ and so this means that unitary operators $U_m(t)$ converge strongly to a unitary operator $U(t)$ on $\mathscr{H}$ since vectors f are dense in $\mathscr{H}$. We proved $U(t)\in\pi(\mathscr{A}^G)''$ and evidently $U(t+t')=U(t)U(t')$.

Remark that by Lemma 3.8, $N(p,b)$ commutes with all $y\in\mathscr{A}_m^G\subset\mathscr{A}_p^G$. Hence $J_p=\sum_{j=1}^{p}(1-\omega(\varphi_j(A)))-2N(p,b)$ commutes with y and so does $U_p(t)$.

Then we have as $p\longrightarrow\infty$

$$\|U(t)\pi(y)f-\pi(y)U(t)f\|\leqq\|(U(t)-U_p(t))\pi(y)f\|+\|\pi(y)(U_p(t)-U(t))f\|\longrightarrow 0.$$

Since vectors f are dense in $\mathscr{H}$, we conclude $U(t)\pi(y)=\pi(y)U(t)$ for $y\in\mathscr{A}_m^G$ hence for $y\in\mathscr{A}^G$. We proved $U(t)\in\pi(\mathscr{A}^G)'$.

We make following calculations

$$|(U(t)f_0, f_0)|=\lim_{m\to\infty}|(U_m(t)f_0, f_0)|=\lim_{m}|\omega(\exp(itJ_m))|$$
$$=\lim|\omega(\exp(it\sum_{j=1}^{m}(\varphi_j(A)-\omega(\varphi_j(A)))))|$$
$$=\lim|\omega(\prod_{j=1}^{m}\exp(it(\varphi_j(A)-\omega(\varphi_j(A)))))|$$
$$=\lim|\prod_{j=1}^{m}\omega(\varphi_j(e_{bb}\exp(-it)+\sum_{c\neq b}e_{cc}\exp(it))\exp(-it\ \omega(\varphi_j(A)))|$$
$$=\lim|\prod_{j=1}^{m}(\cos t+i\cdot\sin t\cdot\omega(\varphi_j)(A)))|$$
$$=\lim_{m\to\infty}\prod_{j=1}^{m}(1-\sin^2 t\cdot(1-\omega(\varphi_j(A))^2))^{\frac{1}{2}}.$$

Since $\omega(\varphi_j(A))\neq\pm1$ for some j, we have for $0<t<\pi$, $|(U(t)f_0, f_0)|<1$. Consequently $U(t)$ is not a scalar for such t. Q.E.D.

5. Sufficient condition for ω^G to be factorial

In this section we show that the state ω^G of $\mathscr{A}^G$ is factorial if $\sum_{j=1}^{\infty}(1-\omega(\varphi_j(A_b))^2)=0$ holds for all $1\leqq b\leqq d$ where A_b is as in section 4.

To show that, we need a Lemma as shown in (7) with a different proof.

Lemma 5.1. Let π be a *-representation of an AF C^*-algebra $\mathscr{A}=\overline{\bigcup\mathscr{A}_m}$. Then $Z(\pi(\mathscr{A})'')=\pi(Z(\mathscr{A}))''$ holds where Z means the center.

Proof. Since $\pi(Z(\mathscr{A}))\subset\pi(\mathscr{A})\cap\pi(\mathscr{A})'$, $\pi(Z(\mathscr{A}))''\subset Z(\pi(\mathscr{A})'')$ holds. For the reverse inclusion we use the fact that the center $Z(\mathscr{A})$ of $\mathscr{A}$ is an AF C^*-algebra $\overline{\bigcup Z(\mathscr{A}_m)}$ (5). Hence it suffices to show that for all integer m, $\pi(\mathscr{A}_m)\cap\pi(\mathscr{A})'\subset\pi(Z(\mathscr{A}))$. Let $\pi(x)$ be in $\pi(\mathscr{A}_m)\cap\pi(\mathscr{A})'$. Now $\mathscr{A}_m=\oplus M(k)$ is a factor decomposition where $M(k)$ is the C^*-algebra of all $h_k\times h_k$ complex matrices and $\{e_{ij}^{(k)};\ 1\leqq i,j\leqq h_k\}$ denotes a matrix unit of the factor $M(k)$. We have $1=\sum_k\sum_i e_{ii}^{(k)}$. Therefore it holds that

$$\pi(x)=\pi(x)\pi(\sum_k\sum_i e_{ii}^{(k)})=\sum_k\sum_i\pi(e_{i1}^{(k)}xe_{1i}^{(k)})=\pi(\sum_{k\cdot i}e_{i1}^{(k)}xe_{1i}^{(k)})$$

since $\pi(x)\in\pi(\mathscr{A})'$. But $\sum_k\sum_i e_{i1}^{(k)}xe_{1i}^{(k)}$ commutes with $\mathscr{A}_m$ by a simple calculation. Hence $\sum_k\sum_i e_{i1}^{(k)}xe_{1i}^{(k)}\in Z(\mathscr{A}_m)\subset Z(\mathscr{A})$ and so $\pi(x)\in\pi(Z(\mathscr{A}))$. the proof is complete. Q.E.D.

The center $Z(\mathscr{A}^G)$ of $\mathscr{A}^G$ is generated by minimal central projections $E_{m,k}=\sum\{f_{\phi\phi};\ \phi\in\Lambda_{m,k}\}$, $m\geqq1$, $k\in K_m$ as $\{E_{m,k};\ k\in K_m\}$ generates $Z(\mathscr{A}_m{}^G)$. Now we give a sufficient condition for ω^G to be factorial.

Theorem 5.2. If $\sum_{b=1}^{d}\sum_{k=1}^{\infty}(1-\omega(\varphi_k(A_b))^2)=0$, then ω^G is factorial.

Proof. When we let $(\pi,\mathscr{H},f_0)$ be the cyclic *-representation of $\mathscr{A}^G$ constructed from ω^G, it suffices to show that by Lemma 5.1, $\pi(Z(\mathscr{A}^G))$ is scalar multiples of the identity operator on $\mathscr{H}$. We shall show that for all integer m $\pi(Z(\mathscr{A}_m{}^G))=C1$ holds. Let E_k; $k\in K_m$ be minimal central projections in $\mathscr{A}_m{}^G$ with $E_k\in M_{m,k}$ (See Theorem 3.1). Put $I_b=\{t;\ \omega(\varphi_t(e_{bb}))=1\}$ for $1\leqq b\leqq d$, $k_b=\#[I_b\cap\{1,2,\dots,m\}]$ and $k_0=(k_1,k_2,\dots,k_d)\in K_m$. Then from the form of E_k we have $\omega(E_k)=\delta_{kk_0}$. Now we shall show $\pi(E_k)=\delta_{kk_0}1$. Since $Z(\mathscr{A}_m{}^G)$ is generated by E_k's, the proof will be then furnished. Put $f=\pi(x)f_0\in\mathscr{H}$ for

$x \in \mathscr{A}_m{}'^G$. Since $E_k \in Z(\mathscr{A}^G)$, we have for all $k \neq k_0$

$$\|\pi(E_k)f\|^2 = \|\pi(E_k x)f_0\|^2 = \|\pi(xE_k)f_0\|^2 = \omega(E_k x^* x E_k) \leqq \|x\|^2 \omega(E_k) = 0.$$

Since such vectors f are dense in $\mathscr{H}$, we have $\pi(E_k)=0$ for $k \neq k_0$. On the other hand we have $\pi(E_{k_0}) = \pi(1 - \sum_{k \neq k_0} E_k) = 1$. Q.E.D.

The author has given a necessary and sufficient condition in order that ω^G is factorial in the case $d=1$ in the previous paper (7) but he does not know yet the condition for ω^G to be factorial when $2 \leqq d \leqq n-1$.

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