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Remarks on Connection Forms

by

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Let M be a differentiable manifold with dim $M=n$ and a local coordinate system (φ, U) such that its local Coordinate $(x^1 \cdots\cdots x^n)$. Let $\{\theta^1, \cdots\cdots \theta^n\}$ be the dual coframes of the field of coordinate frames $\{E_1, \cdots\cdots E_n\}$ on U. For a field of coframes $\{\tilde{\theta}^1 \cdots \tilde{\theta}^n\}$. on $\{E_1, \cdots\cdots E_n\}$ is the dual frames of $\tilde{\theta}^1, \cdots \tilde{\theta}^n\}$.

In this paper we assert that

(a) if M is Riemannian then there exists a unique determined set of $n^2 c^\infty$ one form $\tilde{\theta}^k{}_j$ such that

$$\text{i) } d\tilde{\theta}^i - \sum_{k=1}^{n} \tilde{\theta} \wedge \tilde{\theta}^i{}_j = 0$$

$$\text{ii) } d\tilde{g}_{ij} = \sum_{k=1}^{n} (\tilde{\theta}_i{}^k \tilde{g}_{kj} + \theta^k{}_j \tilde{g}_{ki}),$$

Where $(\tilde{E}_i, \tilde{E}_j) = \tilde{g}_{ij}$ and d the exterior differentiation (theorem 1)

(b) the inverse of (a) is true (theorem 2)

Let M be a Riemannian manifold with its connection ∇, we put

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma_{ij}{}^k E_k, \quad \nabla_{\tilde{E}_i} \tilde{E}_j = \sum_{k=1}^{n} \tilde{\Gamma}_{ij}{}^k \tilde{E}_k.$$

Proposition. 1. The n^3 c^∞ function $\Gamma_{ij}{}^k$ are completly and uniquely determined by n^3 c^∞ function $\Gamma_{ij}{}^k$, and the inverse is true.

Proof. In order to prove our assertion we shall put

$$E_i = \sum_{j=1}^{n} a_{ij}(x) \tilde{E}_j,$$

then each a_{ij}: $U \to \boldsymbol{R}$ is a c^∞ function.

By the properties of ∇ we have the following:

$$\nabla_{E_i} E_j = \sum_{l,m} (a_{il} a_{jm} \nabla_{\tilde{E}} \tilde{E}_m + a_{il} \tilde{E}_l (a_{jm}) \tilde{E}_m)$$

$$=\sum_{s=1}^{n}\left(\sum_{l,m} a_{il}\, a_{jm}\tilde{\Gamma}_{lm}{}^{s}+\sum_{l} a_{il}\, \tilde{E}_l(a_{js})\right) \tilde{E}_s.$$

On the other hand,

$$\nabla_{E_i} E_j=\sum_{k=1}^{n} \Gamma_{ij}{}^{k} E_k=\sum_{k,s} a_{ks}\, \Gamma_{ij}{}^{k} \tilde{E}_s=\sum_{s=1}^{n}\left(\sum_{k=1}^{n} a_{ks}\, \Gamma_{ij}{}^{k}\right) \tilde{E}_s.$$

Therefore, it follows that

$$\sum_{k=1}^{n} a_{ks}\, \Gamma_{ij}{}^{k}=\sum_{l,m} a_{il}\, a_{jl}\, \tilde{\Gamma}_{lm}{}^{s}+\sum_{l} a_{il}\, \tilde{E}_l(a_{js}) \quad (※).$$

Note that since for all $x \in \varphi\ (U)$

$$\begin{pmatrix} a_{11}(x) \cdots\cdots\cdots a_{1n}(x) \\ \vdots \qquad\qquad \vdots \\ a_{n1}(x) \cdots\cdots\cdots a_{nn}(x) \end{pmatrix}$$

is not singular, we have the invese $(a^{ij}(x))$ of $(a_{ij}(x))$ and thus

$$\tilde{E}=\sum_{j=1}^{n} a^{ij} E_j.$$

Therefore

$$\tilde{E}_l(a_{js})=\sum_{j=1}^{n} a^{lj}(x)\, E_j(a_{js})$$

which makes sense.

From (※) above

$$a_{11}\Gamma_{ij}{}^{1}+a_{21}\, \Gamma_{ij}{}^{2}+\cdots+a_{n1}\, \Gamma_{ij}{}^{n}=\sum_{l,m} a_{il}\, a_{jm}\, \tilde{\Gamma}_{lm}{}^{1}+\sum_{l} a_{il}\, \tilde{E}_l(a_{j1})$$

$$a_{1n}\Gamma_{ij}{}^{1}+a_{2n}\, \Gamma_{ij}{}^{2}+\cdots+a_{nn}\, \Gamma_{ij}{}^{n}=\sum_{l,m} a_{il}\, a_{jm}\, \tilde{\Gamma}_{lm}{}^{n}+\sum_{l} a_{il}\, \tilde{E}_l(a_{jn})$$

As before, since the $n \times n$ matrix

$$\begin{pmatrix} a_{11} \cdots\cdots\cdots a_{n1} \\ \vdots \qquad\qquad \vdots \\ a_{1n} \cdots\cdots\cdots a_{nn} \end{pmatrix}$$

is not singular, we have a unique solution

$$\{\Gamma_{ij}{}^k \mid k=1,\cdots\cdots n\}.$$

That is, $\{\Gamma_{ij}{}^k \mid 1\leq i,\ j,\ k\leq n\}$ is uniqudly determined by n^3 c^∞ functions $\{\Gamma_{ij}{}^k \mid 1\leq i,\ j,\ k\leq n\}$. The inverse is proved by the same way as above.

With the above notations we define

$$\tilde{\theta}_j{}^k=\sum_{l=1}^{n}\tilde{\Gamma}_{lj}{}^k\,\theta^l,$$

then the n^3 functions $\tilde{\Gamma}_{lj}{}^k$ are C^∞.

For a C^∞ vector field $X(\in *(M))$

$$\nabla_x\tilde{E}_j=\sum_{k=1}^{n}\tilde{\theta}_j{}^k(X)\,\tilde{E}_i,$$

Where $\nabla_{\tilde{E}_i}\tilde{E}_j=\sum_{k=1}^{n}\tilde{\Gamma}_{ij}{}^k\,\tilde{E}_k$ $(1\leq i,\ j,\ k\leq n)$. (Note that $[\tilde{E}_i,\ \tilde{E}_j]\neq 0$ implies $\tilde{\Gamma}_{ij}{}^k\neq\tilde{\Gamma}_{ji}{}^k$, where $[,]$ is the bracket product.)

Theorem 1. Under the above situation we have the follwing.

(i) $d\tilde{\theta}_i=\sum_{k=1}^{n}ax^k\wedge\dfrac{\partial\tilde{\theta}^i}{\partial x^k}=\sum_{j=1}^{n}\tilde{\theta}^j\wedge\tilde{\theta}_j{}^i.$

(ii) $d\tilde{g}_{ij}=\sum_{k=1}^{n}\dfrac{\partial\tilde{g}_{ij}}{\partial x^k}\,dx^k=\sum_{k=1}^{n}(\tilde{\theta}_i{}^k\,g_{kj}+\tilde{\theta}_i{}^k\,g_{ki}),$

where $\tilde{g}_{ij}=(\tilde{E}_i,\ \tilde{E}_j)$. (Note that d is exterior differentiation).

Proof: Let us put

$$E_i=\sum_{j=1}^{n}a_{ij}(x)\,\tilde{E}_j\ \ (1\leq i\leq n),$$

then $\{a_{ij}|i,j,\leq n\}$ are C^∞ functions and the $n\times n$ matrix (a_{ij}) is a non—singular matrix at each $x\in\varphi(U)\subset R^n$. It follows that

$$\tilde{E}_i=\sum_{j=1}^{n}a^{ij}\,E_i\quad(1\leq i\leq n),$$

where $(a_{ij})^{-1}=(a^{ij})$.

In this case we can easily prove that

$$\theta^i=\sum_{j=1}^{n} a^{ji}\,\tilde{\theta}^i,\quad \tilde{\theta}^i=\sum_{j=1}^{n} a_{ji}\,\theta^j\ (1\leq i\leq n).$$

$$\begin{aligned}(\mathrm{i})\ :d\tilde{\theta}^i&=\sum_{k=1}^{n}\theta^k\wedge E_k(\tilde{\theta}_i)=\sum_{k=1}^{n}\theta^k\wedge E_k(\sum_{j=1}^{n} a_{ji}\,\theta^j)\\&=\sum_{k,j}\theta^k\wedge E_k(a_{ji})\,\theta^j\\&=\sum_{k,j}E_k(a_{ji})\,\theta^k\wedge\theta^j\\&=\sum_{m,s}E_m(a_{si})\,\theta^m\wedge\theta^s\\&=-\sum_{m,s}E_m(a_{si})\,\theta^s\wedge\theta^m.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{n}\tilde{\theta}^i\wedge\tilde{\theta}_j{}^i&=\sum_{j=1}^{n}(\tilde{\theta}^j\wedge\sum_{j=1}^{n}\tilde{\Gamma}_{lj}{}^i\,\tilde{\theta}^l)\\&=\sum_{j,l}(\sum_{s=1}^{n}a_{sj}\,\theta^s\wedge\tilde{\Gamma}_{lj}{}^i\sum_{m=1}^{n}a_{ml}\,\theta^m)\\&=\sum_{m,s}(\sum_{j,l}a_{ml}\,a_{sj}\,\tilde{\Gamma}_{lj}{}^i)\theta^s\wedge\theta^m.\end{aligned}$$

By (※) in the proof of propositon 1,

$$\begin{aligned}\sum_{t=1}^{n}a_{ti}\,\Gamma_{ms}{}^t&=\sum_{l,s}a_{ml}\,a_{sj}\,\tilde{\Gamma}_{lj}{}^i+\sum_{l=1}^{n}a_{ml}\,\tilde{E}_l(a_{si})\\&=\sum_{l,j}a_{ml}\,a_{sj}\,\tilde{\Gamma}_{lj}{}^i+E_m(a_{si}).\end{aligned}$$

Thus, we have,

$$\begin{aligned}\sum_{j=1}^{n}\tilde{\theta}^j\wedge\tilde{\theta}_j{}^i&=\sum_{m,s}(\sum_{t=1}^{n}a_{ti}\,\Gamma_{ms}{}^t-E_m(a_{si}))\theta^s\wedge\theta^m\\&=\sum_{t=1}^{n}a_{ti}(\sum_{m,s}\Gamma_{ms}{}^t\,\theta^s\wedge\theta^m)-\sum_{m,s}E_m(a_{si})\,\theta^s\wedge\theta^m\end{aligned}$$

Since $\Gamma_{ms}{}^t=\Gamma_{sm}{}^t$ and $\theta^s\wedge\theta^m=-\theta^m\wedge\theta^s$ the first term of the left side in the last expression is zero, we have,

$$\sum_{j=1}^{n} \tilde{\theta}^{j} \wedge \tilde{\theta}_{j}{}^{i} = -\sum_{m,s} E_m(a_{si})\, \theta^s \wedge \theta^m.$$

Therefore our assertion was proved.

(ii) $d\tilde{g}_{ij} = \sum_{k=1}^{n} (E_k(\tilde{E}_i, \tilde{E}_j))\theta^k = \sum_{k=1}^{n} \{(\nabla_{E_k}\tilde{E}_i, \tilde{E}_j) + (\tilde{E}_i, \nabla_{E_k}\tilde{E}_j)\}\theta^k$

(by the properties of the connection ∇)

$$= \sum_{k=1}^{n} (\sum_{l=1}^{n} a_{kl}(\nabla_{\tilde{E}l}\tilde{E}_i, \tilde{E}_j) + \sum_{l=1}^{n} a_{kl}(\tilde{E}_i, \nabla_{\tilde{E}l}\tilde{E}_j))\theta^k$$

$$= \sum_{k=1}^{n} (\sum_{l,s} a_{kl}\, \Gamma_{li}{}^{s}\tilde{g}_{si} + \sum_{l,s} a_{kl}\, \tilde{\Gamma}_{lj}{}^{s}\tilde{g}_{is})\, (\sum_{t=1}^{n} a^{tk}\tilde{\theta}^t)$$

$$= \sum_{l,s,t} (\sum_{k=1}^{n} a_{kl}\, a^{tk}\, \tilde{\Gamma}_{li}{}^{s}\, \tilde{g}_{sj} + \sum_{k=1}^{n} a_{kl}\, a^{tk}\, \tilde{\Gamma}_{lj}{}^{s}\tilde{g}_{is})\tilde{\theta}^t$$

$$= \sum_{l,s} (\tilde{\Gamma}_{li}{}^{s}\tilde{\theta}^l)\tilde{g}_{sj} + \tilde{\Gamma}_{lj}{}^{s}\tilde{\theta}^l\tilde{g}_{is})$$

$$= \sum_{s=1}^{n} ((\sum_{l=1}^{n}\tilde{\Gamma}_{li}{}^{s}\tilde{\theta}^l)\tilde{g}_{si} + (\sum_{l=1}^{n}\tilde{\Gamma}_{lj}{}^{s}\tilde{\theta}^l)\tilde{g}_{is})$$

$$= \sum_{k=1}^{n} (\tilde{\theta}_i{}^k\tilde{g}_{kj} + \tilde{\theta}_j{}^k\, \tilde{g}_{ik}). \qquad ///$$

For a while, by M We mean a differentable manifold with dim $M=n$. Let (φ, U) be a local coordinate system with local coordinate $(x^1, \cdots\cdots x^n)$ and the field of coordinate frames $\{E_1, \cdots E_n\}$, and let $\{\theta^1, \cdots \theta^n\}$ be the dual coframes of $\{E_1, \cdots E_n\}$. We consist a field of coframes $\{\tilde{\theta}^1, \cdots \tilde{\theta}^n\}$ defined on U and its dual frames $\{\tilde{E}_1, \cdots \tilde{E}_n\}$. We suppose that there are n^2 C^∞ one-formes $\tilde{\theta}_j{}^k (1 \leqq j,\ k \leqq n)$ which satisfy conditions (i) and

(ii) in Theorem 1, and define the following;

(iii) For each C^∞ vector field X defined on M

$$\nabla_x \tilde{E}_j = \sum_{k=1}^{n} \tilde{\theta}_j{}^k(x)\tilde{E}_k,$$

where ∇ is a $C^\infty(M)$-linear map with respect to X.

That is, for $f \in C^\infty(M)$, $\nabla_{fx}\tilde{E}_j = f\nabla_x\tilde{E}_j$.

(iv) For $f \in C^\infty(M)$

$$\nabla_x(fY) = (xf)Y + f\nabla_x Y.$$

Then, we have the following (Theorem).

Theorem 2 Under the above circumstance:

(a) For X, $Y \in *(M)$

condition (i) in Theoem 1 $\Leftrightarrow$ $[X,Y]=\nabla_x Y-\nabla_y X$

(b) For X, Y, $Z \in *(M)$

condition (ii) in Theorem 1 $\Leftrightarrow$ $X(Y,Z)=(\nabla_x Y,Z)+(Y,\nabla_x Z)$.

Proof. If we use $df(x)=xf(f\in C^\infty(M),\ x\in *(M))$ then it is easy to check that for each one form W

$$dW(X,Y)=Xw(Y)-Yw(X)-w([X,Y]) \quad (※※)$$

Where X, $Y\in *(M)$

(a): For $X,Y\in *(M)$ we put $X=\sum_{i=1}^{n}\alpha^i(x)\tilde{E}_i$ and $Y=\sum_{i=1}^{n}\beta^i(x)\tilde{E}_i$, then α^i and β^i are C^∞ functions on $\varphi(U)\subset R^n$ fore $i=1,\cdots n$.

($\Rightarrow$) By (※※) $\sum_{i=1}^{n}(d\tilde{\theta}^i(X,Y))\tilde{E}_i=\sum_{i=1}^{n}(X(\beta^i)-(Y(\alpha^i)-\tilde{\theta}^i([X,Y]))\tilde{E}_i$ using $[X,Y]=XY-YX$ We have the following.

$$\sum_{i=1}^{n}(\tilde{\theta}^i([x,Y]))\tilde{E}_i=\sum_{i=1}^{n}(x(\beta^i)\tilde{E}_i-Y(\alpha^i)\tilde{E}_i)=xY-Yx=[x,Y].$$

Thus,

$$\sum_{i=1}^{n}(d\tilde{\theta}^i(X,Y))\tilde{E}_i=\sum_{i=1}^{n}(X(\beta^i)-Y(\alpha^i))\tilde{E}_i-[X,Y].$$

On the other hand,

$$\sum_{i=1}^{n}(\sum_{j=1}^{n}\tilde{\theta}^j\wedge\tilde{\theta}_j{}^i(X,Y))\tilde{E}_i=\sum_{j=1}^{n}\tilde{\theta}^j(x)\sum_{i=1}^{n}\tilde{\theta}_j{}^i(Y)\tilde{E}_i-\sum_{j=1}^{n}\tilde{\theta}^j(Y)\sum_{i=1}^{n}\tilde{\theta}_j{}^i(X)\tilde{E}_i$$
$$=\sum_{j=1}^{n}\alpha^j\nabla_Y\tilde{E}_j-\sum_{j=1}^{n}\beta^j\nabla_X\tilde{E}_j.$$

Condition (i) in Theorem 1 implies that

$$\sum_{j=1}^{n}(X(\beta^j)-Y(\alpha^j))\tilde{E}_j-[X,Y]=\sum_{j=1}^{n}(\alpha^j\nabla_Y\tilde{E}_j-\beta^j\nabla_X\tilde{E}_j),$$

and thus

$$[X,Y]=\sum_{j=1}^{n}(X(\beta^j)\tilde{E}_j+\beta^j\nabla_X\tilde{E}_j)-\sum_{j=1}^{n}(Y(\alpha^j)\tilde{E}_j+\alpha^j\nabla_Y\tilde{E}_j)$$
$$=\nabla_X Y-\nabla_Y X$$

($\Leftarrow$): By (※※) $d\tilde{\theta}^i(X,Y)=X\tilde{\theta}^i(Y)-Y\tilde{\theta}^i(X)-\tilde{\theta}^i([X,Y])$

$$=X(\beta^i)-Y(\alpha^i)-\tilde{\theta}^i(\nabla_X Y-\nabla_Y X)$$

$$=X(\beta^i)-Y(\alpha^i)-\tilde{\theta}^i(\sum_{j=1}^n(X(\beta^j)\tilde{E}_j+\beta^j\sum_{k=1}^n\tilde{\theta}_j{}^k(X)\tilde{E}_k)$$

$$-\sum_{j=1}^n(Y(\alpha^j)\tilde{E}^j-\alpha^j\sum_{k=1}^n\tilde{\theta}_j{}^k(Y)\tilde{E}_k$$

$$=X(\beta^i)-Y(\alpha^i)-X(\beta^i)+Y(\alpha^i)$$

$$-\sum_{j=1}^n(\beta^i\tilde{\theta}_j{}^i(X)-\alpha^i\tilde{\theta}_j{}^i(Y))$$

$$=\sum_{j=1}^n(\alpha^j\tilde{\theta}_j{}^i(Y)-\beta^j\tilde{\theta}_j{}^i(Y))$$

$$=\sum_{j=1}^n\tilde{\theta}^j\wedge\tilde{\theta}_j{}^i(X,Y)$$

Thus, we get $d\tilde{\theta}^i=\sum_{j=1}^n\tilde{\theta}\wedge\tilde{\theta}_j{}^i$.

(b): For $X,Y,Z\in*(M)$ we shall put

$$X=\sum_{i=1}^n\alpha^i(x)E_i,\ Y=\sum_{i=1}^n\beta^i(x)\tilde{E}_i,\ Z=\sum_{i=1}^n\gamma^i(x)\tilde{E}_i.$$

($\Rightarrow$): Note that

$$d\tilde{g}_{ij}(x)=\sum_{k=1}^n\frac{\partial\tilde{g}_{ij}}{\partial x^k}\theta^k(x)=\sum_{k=1}^n\alpha^k\frac{\partial\tilde{g}_{ij}}{\partial x^k}=X(\tilde{g}_{ij})=(\sum_{k=1}^n\alpha^k\frac{\partial}{\partial x^k})\tilde{g}_{ij}.$$

Using this we calculate as follows.

$$(\nabla_X Y,Z)+(Y,\nabla_X Z)=(\sum_{i=1}^n(X(\beta^i)\tilde{E}_i+\beta^i\nabla_X\tilde{E}_i),\ \sum_{j=1}\gamma^j\tilde{E}_j)$$

$$+(\sum_{i=1}^n\beta^i\tilde{E}_i,\ \sum_{j=1}^n(X(\gamma^j)\tilde{E}_j+\gamma^j\nabla_X\tilde{E}_j)$$

$$=\sum_{i,j}(X(\beta^i)\gamma^j\tilde{g}_{ij}+\beta^i\gamma^j\sum_{k=1}^n\tilde{\theta}_i{}^k(X)\tilde{g}_{kj}+\beta^iX(\gamma^j)\tilde{g}_{ij}$$

$$+\beta^i\gamma^j\sum_{k=1}^n\tilde{\theta}_j{}^k\tilde{g}_{ik})$$

$$=\sum_{i,j}(X(\beta^i)\gamma^j+\beta^iX(\gamma^j))\tilde{g}_{ij}+\sum_{i,j}\beta^i\gamma^j\sum_{k=1}^n(\tilde{\theta}_i{}^k(x)\tilde{g}_{ik}$$

$$+\tilde{\theta}_i^k(x)\tilde{g}_{ik})$$

$$=\sum_{i,j}(X(\beta^i)\gamma^j+\beta^i X(\gamma^j)\tilde{g}_{ij}+\sum_{i,j}\beta^i\gamma^j d\tilde{g}_{ij}(x)$$

$$=\sum_{i,j}[(X(\beta^i)\gamma^j+\beta^i X(\gamma^j))\tilde{g}_{ij}+\beta^i\gamma^j X(\tilde{g}_{ij})]$$

$$=X(Y,Z).$$

($\Leftarrow$): From the above descriptions and $X(Y,Z)=(\nabla_X Y,Z)+(Y,\nabla_X Z)$ We have the following;

$$\sum_{i,j}[(X(\beta^i)\gamma^j+\beta^i X(\gamma^j))\tilde{g}_{ij}+\beta^i\gamma^j d(\tilde{g}_{ij})(x)]$$

$$=\sum_{i,j}[(X(\beta^i)\gamma^j+\beta^i X(\gamma^j))\tilde{g}_{ij}+\beta^i\gamma^j\sum_{k=1}^{n}(\tilde{\theta}_i^k\tilde{g}_{ki})(x)]$$

Therefore we have $d\tilde{g}_{ij}=\sum_{k=1}^{n}(\tilde{\theta}_i^k\tilde{g}_{kj}+\tilde{\theta}_j^k\tilde{g}_{ki})$

From now on, a differentiable manifold $M(\dim M=n)$ is covered by a local coordinate system (φ,U) $(M=U)$.

We shall use the above notations.

Corollary 3. If M is a Riemannian manifold then there exists a uniquely determined set of n^2C^∞ oneforms $\tilde{\theta}_j^k(1\leq j,\ k\leq n)$ satisfying (i) and (ii) Theorem 1 conversely, if M has a set of n^2C^∞ one forms $\tilde{\theta}_j^k(1\leq j,k\leq n)$ satisfying (i) and (ii) in Theorem 1 then M is a Riemannian manifold.

Poof. Let ∇ be the Riemannian connection of M. Define

$\nabla_{\tilde{E}_i}\tilde{E}_j=\sum_{k=1}^{n}\tilde{\Gamma}_{ij}^k\tilde{E}_k$, then by proposition 1 the n^3 C^∞ functions

$\tilde{\Gamma}_{ij}^k(1\leq i,j,k\leq n)$ are uniquely determined using $\nabla(i.e.,\Gamma_{ij}^k)$.

Define

$$\tilde{\theta}_j^k=\sum_{l=1}^{n}\tilde{\Gamma}_{lj}^k\tilde{\theta}^l,$$

Then the n^2 C^∞ one forms $\tilde{\theta}_j^k(1\leq j,k\leq n)$ satisfy (i) and (ii) is Theorem 1 by Theorem 1

Conversely, we put $\tilde{\Gamma}_{ij}^k=\tilde{\theta}_j^k(\tilde{E}_i)$, and define

$$\nabla : *(M) X *(M) \rightarrow *(M)$$

by (iii) and (iv) above. Then we have $\nabla^{\tilde{E}}{}_i \tilde{E}_j = \sum_{k=1}^{n} \tilde{\Gamma}_{ij}{}^k \tilde{E}_k (1 \le i, j \le n)$,

If we put $\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma^k{}_{ij} E_k (1 \le i, j \le n)$, then by proposition The n^3 C^∞ functions $\Gamma_{ij}{}^k$ are uniquely determined using $n^3 C^\infty$ functions $\tilde{\Gamma}_{ij}{}^k$. By theorem 3 we have the following.

$$0 = [E_i, E_j] = \nabla_{E_i} E_j - \triangle_{E_j} E_i \Rightarrow \Gamma_{ij}{}^k = \Gamma_{ji}{}^k (1 \le i, j, k \le n)$$

$$E_k(E_i, E_j) = (\nabla_{E_k} E_i, E_j) + (E_i, \nabla_{E_k} E_j) \Rightarrow E_k g_{ij} = \sum_{k=1}^{n} (\Gamma_{ki}{}^s g_{sj} + \Gamma_{kj}{}^s g_{si}),$$

where $(E_i, E_j) = g_{ij}$.

Thus the ∇ defined by $n^3 C^\infty$ functions $\Gamma_{ij}{}^k (1 \le i, j, k \le n)$ is the unique Riemannian connection on M which are defined by $\tilde{\theta}_i{}^k (1 \le j, k \le n)$ ([1]). Accordingly, M is a Riemannian manifold.

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