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Maximal Ideal Spaces and B^*-Algebras

by

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1. Introduction

Interest in the topic of normed space in Functional analysis has escalated in the 1950s. Therefore, during the nearly two decades there has been a remarkable developments in Functional analysis, stretching its study to Banach space, Banach algebra, and C^*-algebra ([2], [8], [9]).

The main core of Banach algebra can largely be divided into Maximal ideal space theory and Spectral theory ([3], [5]).

It is, therefore, linear operator theory that is vital part of Functional analysis ([6], [7]).

In this view of Functional analysis, this paper deals with maximal ideal space and B^*-algebra. It can, in more details, be described as follows.

§2 contains the proof of proposition 2.2 which is a little properties of Banach algebra's element, explaining the required terminology to grasp §3 and §4.

§3 gives proofs of theorem 3.1 and corollary 3.2 which is a part of the main theory in this paper. That is, if X is a compact and Hausdorff space, then $C(X)$ becomes commutative Banach algebra.

Then, Theorem 3.1, it is argued;

(i) $C(X)$ is semisimple

(ii) The maximal ideal space of $C(X)$, $\triangle$ is homeomorphic to X.

In that case, corollary 3.2 shows the relation of one-to-one correspondence under a condition, which is

$$\left\{\begin{array}{l}\text{The set of all closed}\\ \text{subsets of } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{The set of all}\\ \text{closed ideals of } C(X)\end{array}\right\}.$$

In §4, we prove Theorem 4.1 which consists of one of the main theorem in this paper. Theorem 4.1 offers three properties of B^*-algebra, proving that disc algebra is a commutative B^*-algebra, in proposition 4.2. In addition, it ie verified, by an example of $*$-algebra in Example 4.3, that if $xy \neq yx$, then $\sigma(x+y) \subset \sigma(x)+\sigma(y)$ and $\sigma(xy) \subset \sigma(x) \cdot \sigma(y)$ don't hold.

§2. Preliminaries

Throughout this paper, by a Banach algebra we mean a Banach algebra which contains the identity e.

Definition 2.1 Let A be a Banach algebra. For each $x \in A$, the *spectrum* $\sigma(x)$ of x is the set of all complex numbers λ such that $\lambda e - x$ is not invertible. The *spectral radius* $\rho(x)$ of x is the number $\rho(x) = \sup\{|\lambda| | \lambda \in \sigma(x)\}$.

It is well-known that for each $x \in A$

1.° $\sigma(x)$ is compact and non-empty,

2.° $\rho(x) = \lim_{n \to \infty} \|x^n\|^{1/n} \leq \|x\|$.

Proposition 2.2 In a Banach algebra A the following hold.

(i) $\forall x, y \in A$ if $e - xy$ is invertible, then so is $e - yx$.

(ii) $\forall x, y \in A, \lambda \in \sigma(xy) \Longrightarrow \lambda \in \sigma(yx)$.

Proof. (i) Suppose that there is an element $z \in A$ such that $(e-xy)z = e$, *i.e.*, $z - xyz = e$. Then,

$$\begin{aligned}(e-yx)\ (e+yzx) &= e+yzx-yx-yxyzx \\ &= e+yzx-yx-y(z-e)x \\ &= e.\end{aligned}$$

Hence, $e+yzx$ is the inverse element of $e-yx$, and thus $e-yx$ is invertible.

(iii) When $\lambda = 0 \in \sigma(xy)$, we assume that yx is invertible.
Then there exists an element $z \in A$ such that

$$z(yx) = (zy)x = e = (yx)z = y(xz).$$

This implies that x and y are invertible in A, and thus xy is invertible. Hence, in this case, yx is not invertible, and $0 \in \sigma(yx)$.

Next, we assume that $\lambda (\neq 0) \in \sigma(xy)$ and that $\lambda e - yx$ is invertible. Then, there exists an element $z \in A$ such that $(\lambda e - yx)z = e$. Therefore, since $yxz = \lambda z - e$, we have the following:

$$\begin{aligned}(\lambda e-xy)(\lambda^{-1}e+\lambda^{-1}xzy)&=e+xzy-\lambda^{-1}xy-\lambda^{-1}xyxzy\\&=e+xzy-\lambda^{-1}xy-\lambda^{-1}x(\lambda z-e)y\\&=e.\end{aligned}$$

That is, $\lambda e-xy$ is invertible. This is a contradiction to our hypothesis $\lambda\in\sigma(xy)$. ///

Let A be a Banach algebra, and let Ω be an open subset of $\boldsymbol{C}$. Put

$$H(\Omega)=\{f:\ \Omega\longrightarrow \boldsymbol{C}\,|\,f \text{ is holomorphic}\}$$

and

$$A_\Omega=\{x\in A\,|\,\sigma(x)\subset\Omega\}.$$

For each $f\in H(\Omega)$ we define the mapping $\tilde{f}:\ A_\Omega\longrightarrow A$
by

$$\tilde{f}(x)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e-x)^{-1}\,d\lambda,$$

where $x\in A_\Omega$ and Γ is any contour that surrounds $\sigma(x)$ in Ω. Put

$$\tilde{H}(\Omega)=\{\tilde{f}\,|\,f\in H(\Omega)\}.$$

Then it is obvious that $H(\Omega)$ and $\tilde{H}(\Omega)$ are algebras under the usual operations.

Lemma 2.3 Under the above notations the map

$$\begin{array}{cccc}\Phi: & H(\Omega) & \longrightarrow & \tilde{H}(\Omega)\\ & \Psi & & \Psi\\ & f & \longmapsto & \tilde{f}\end{array}$$

is an algebra isomorphism.

Proof. It is clear that Φ is one-to-one and linear. So, it suffices to prove that Φ is multiplicative. That is, we shall prove that if $h(\lambda)=f(\lambda)\cdot g(\lambda)(f,g\in H(\Omega)$ and $\lambda\in\Omega)$ then $\tilde{h}(x)=\tilde{f}(x)\cdot\tilde{g}(x)$ for all $x\in A_\Omega$ (note that $h\in H(\Omega)$).

Consider a rational function

$$R(\lambda)=P(\lambda)+\sum_{m,k}C_{m,k}(\lambda-\alpha_m)^{-k}$$

with poles at the points $\alpha_m(\alpha_m,\ C_{m,k}\in\boldsymbol{C})$, where $P(\lambda)$ is a polynomial in λ with coefficients in $\boldsymbol{C}$. If we put

$$R(x)=P(x)+\sum_{m,k}C_{m,k}(x-\alpha_m e)^{-k},$$

then we see that

$$\tilde{R}(x)=\frac{1}{2\pi i}\int_{\Gamma}R(\lambda)\,(\lambda e-x)^{-1}\,d\lambda, \quad\cdots\cdots(2-1)$$

where $R(\lambda)\in H(\Omega)$, $x\in A_{\Omega}$ and Γ surrounds $\sigma(x)$ in Ω ([1], [12]). In particular, if $\alpha\notin\Omega$ ($\alpha\in C$), then

$$\frac{1}{2\pi i}\int_{\Gamma}(\alpha-\lambda)^{n}\,(\lambda e-x)^{-1}\,d\lambda=(\alpha e-x)^{n} \quad\cdots\cdots(2-2)$$

for $n=0, \pm1, \pm2, \cdots\cdots$ ([12]). Hence, if f and g are rational functions by (2—1) and (2—2) we immediately have

$$\tilde{h}(x)=\tilde{f}(x)\cdot\tilde{g}(x).$$

By the Runge's theorem, for each holomorphic function f there exists rational functions f_n such that $f_n\to f$ uniformly on compact subset of Ω. Therefore,

$$\tilde{h}_n(x)=\tilde{f}_n(x)\cdot\tilde{g}_n(x)\longrightarrow\tilde{f}(x)\cdot\tilde{g}(x),$$

where g_n is a rational function such that $g_n\longrightarrow g$.
Since $h_n(x)\longrightarrow h(x)$, we have $\tilde{h}(x)=\tilde{f}(x)\cdot\tilde{g}(x)$. ///

Under the above notations there is the *spectral mapping theorem* such as:

3.° for each $x\in A_{\Omega}$ and $f\in H(\Omega)$, $\sigma(\tilde{f}(x))=f(\sigma(x))$ ([1], [4]).

Definition 2.4 Let A be a Banach algebra. A linear functional ϕ: $A\to C$ is called a *complex homomorphism* of A if for any $x, y\in A$

$$\phi(xy)=\phi(x)\,\phi(y).$$

The set $\triangle$ of all complex homomorphisms of A is called the *maximal ideal space* of A. For each $x\in A$ we define

$$\hat{x}:\ \triangle\longrightarrow C$$

by $\hat{x}(h)=h(x)$ for each $h\in\triangle$. We put

$$\hat{A}=\{\hat{x}\,|\,x\in A\}.$$

Then, the maximal ideal space $\triangle$ of A with the weak topology induced from $\hat{A}$ (i.e., the Gelfand topology) is a compact Hausdorff space ([12]). (note that $\hat{x}$ is a bounded linear map).

We put

$$Rd(A)=\{x\in A \mid \forall h\in\triangle,\ h(x)=0\},$$

and call it the *radical* of A (since each maximal ideal of A is the kernel of an element $h\in\triangle$

$$Rd(A)=\bigcap_{\mathfrak{M}}\mathfrak{M},\ (\mathfrak{M} \text{ is a maximal ideal of } A)).$$

In particular, if $Rd(A)=\{0\}$, then A is said to be *semisimple.*

Lemma 2.5 For commutative Banach algebra A and B, let $\psi: B\longrightarrow A$ be an algebra homomorphism. If A is semi-simple, then ψ is continuous.

Proof. By the closed graph theorem ([1], [14], [15]), it suffices to prove that $\{(b,\psi(b))\mid b\in B\}$ is a closed subset of $B\times A$. Thus, we shall prove that if $(b_n,\psi(b_n))\longrightarrow(b,a)\in B\times A$ then $(b,a)\in\{(b,\ \psi(b))\mid b\in B\}$ which is equivalent to $\psi(b)=a$.

Let $\triangle_B$ and $\triangle_A$ be maximal ideal spaces of B and A, respectively.
For a fixed $h\in\triangle_A$, define ϕ: $B\longrightarrow \boldsymbol{C}$ by $\phi=h\circ\psi$.
It follows that ϕ is a complex homomorphism.
Thus, for each $b\in B$ with $\|b\|=1$ and $\varepsilon>0$, we have

$$\left|\phi\left(\frac{b}{1+\varepsilon}\right)\right|<1$$

([12]). That is,

$$|\phi(b)|<1+\varepsilon\Longrightarrow\forall b \text{ with } \|b\|=1,\ |\phi(b)|\leq 1.$$

Hence ϕ is a bounded linear functional, that is, ϕ is continuous.
Therefore

$$\begin{aligned}h(a)&=\lim_{n\to\infty} h(\psi(b_n))=\lim_{n\to\infty}\phi(b_n)=\phi(b)\\&=h(\psi(b)),\end{aligned}$$

where $h\in\triangle_A$. Thus,

$$\begin{aligned}h(a-\psi(b))=0&\Longrightarrow a-\psi(b)\in Rd(A)=\{0\}\\&\Longrightarrow a=\psi(b).\end{aligned}$$

///

Let A be a commutative Banach algebra. Then,

4.° for every $x\in A$, $\{\hat{x}(h)\mid h\in\triangle\}=\sigma(x)$ and $\|\hat{x}\|_\infty=\rho(x)$, where $\triangle$ is the maximal ideal space of A and $\|\hat{x}\|_\infty = \sup_{h\in\triangle} |\hat{x}(h)|$ ([1]).

Lemma 2.6 Let A be a commutative Banach algebra.
Then, for any $x, y \in A$,

$$\sigma(x+y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x) \cdot \sigma(y).$$

Proof. By 4°, we have

$$\sigma(x+y) = (x+y)^\wedge(\triangle), \quad \sigma(xy) = (xy)^\wedge(\triangle),$$

where $\triangle$ is the maximal ideal space of A. Since

$$(x+y)^\wedge(\triangle) \subseteq \hat{x}(\triangle) + \hat{y}(\triangle),$$
$$(xy)^\wedge(\triangle) \subseteq \hat{x}(\triangle) \cdot \hat{y}(\triangle),$$

Our assertion is clear. ///

Definition 2.7 For a Banach algebra A, a mapping $*: A \longrightarrow A$ defined by $x \mapsto x^*$ is called an *involution* on A if it satisfies the following properties:

(1) $(x+y)^* = x^* + y^*$ $(x, y \in A)$

(2) $(\lambda x)^* = \bar{\lambda} x^*$ $(\lambda \in \boldsymbol{C},\ x \in A)$

(3) $(xy)^* = y^* x^*$ $(x, y \in A)$

(4) $x^{**} = x$ $(x \in A)$.

Furthermore, for each $x \in A$ if $\|xx^*\| = \|x\|^2$ holds, then A is called a *B^*-algebra.*

Let A be a Banach algebra with an involution $*$.
An element $x \in A$ is said to be *Hermitian* if $x = x^*$, and if $xx^* = x^*x$, then x is said to be *normal*.
A Hermitian element $x \in A$ is said to be *positive*, written $x \geq 0$, if $\sigma(x) \subset [0, \infty)$.
A linear functional

$$F: A \longrightarrow \boldsymbol{C}$$

is said to be *positive* if for every $x \in A$ $F(xx^*) \geq 0$.

Let A be a commutative B^*-algebra with its maximal ideal space $\triangle$. Then, it is well-known that

5.° $\forall h \in \triangle$ and $\forall x \in A$, $h(x^*) = \overline{h(x)}$.

Accordingly, in our case

$$x \text{ is hermitian} \Longleftrightarrow \{\hat{x}(h) \mid h \in \triangle\} \subset \boldsymbol{R}.$$

3. Maximal Ideal Spaces

Let X be a compact Hausdorff space, and let $C(X)$ be the set of all continuous functions from X to $\boldsymbol{C}$. Then, with the norm

$$\|f\|=\sup_{x\in X}|f(x)| \qquad (\forall f\in C(X))$$

$C(X)$ is a commutative Banach algebra.

Theorem 3.1 Let $\triangle$ be the maximal ideal space of $C(X)$. Then $\triangle$ is homeomorphic to X and $C(X)$ is semisimple.

Proof. Define

$$\phi:\ X \longrightarrow \triangle \text{ by } \phi(x)=h_x\in\triangle$$

and h_x: $C(X)\longrightarrow \boldsymbol{C}$ is defined by $h_x(f)=f(x)$ for all $f\in C(X)$. It is easy to prove that for all $x\in X$ h_x is a complex homomorphism of $C(X)$.

For $x\neq y$ in X, assume that $h_x=h_y$. Then

$$\forall f\in C(X),\ h_x(f)=h_y(f)\Longrightarrow f(x)=f(y).$$

But, we can make a continuous function $g\in C(X)$ such that $g(x)\neq g(y)$ as follows. Since X is compact and Hausdorff, it is a normal space. Thus, the Urysohn's theorem holds on X. That is, there is a continuous function g: $X\longrightarrow \boldsymbol{C}$ such that $g(x)=0$ and $g(y)=1$. In consequence, ϕ is an injective map.

In fact, ϕ is surjective. That is, for each $h\in\triangle$ there exists a unique element $x\in X$ such that $h=h_x$.
To prove this we assume that our assertion is fail.
Then, there is an element $h\in\triangle$ such that $h\neq h_x$ for all $x\in X$.
In this case, there is at least one element f in the kernel of h (Ker h) such that $f(x)\neq 0$ for a given point $x\in X$. For we assume that

$$\forall f\in \text{Ker } h,\ f(x)=0,$$

then Ker $h\subseteq$Ker h_x. Since Ker h and Ker h_x are maximal ideals of $C(X)$, it follows that Ker $h=$Ker h_x. Since the Banach algebra (every maximal ideal is closed)

$$C(X)/\text{Ker } h=C(X)/\text{Ker } h_x$$

is a field, it is isomorphic to $\boldsymbol{C}$, by the Gelfand-Mazur theorem. Since a complex homomorphism maps the identity to 1 and it is $\boldsymbol{C}$-linear, it is obvious that

$$\bar{h}=\bar{h}_x:\ C(X)/\mathrm{Ker}\ h \xrightarrow{\ \cong\ } \boldsymbol{C},$$

where $\bar{h}$ and $\bar{h}_x$ are induced from h and h_x, respectively.
Consequently, we have $h=h_x$. Therefore, there is at least one element $f\in\mathrm{Ker}\ h$ such that $f(x)\neq 0$ for each point $x\in X$.

Take $f_x\in\mathrm{Ker}\ h$ such that $f_x(x)\neq 0$. Since f_x is continuous, there exists an open neighborhood U_x of x such that $f_x|U_x\neq 0$. Then

$$X=\bigcup_{x\in X} U_x,$$

and there is a finite set $\{U_{x_1},\cdots\cdots,U_{x_n}\}\subset\{U_x\,|\,x\in X\}$, because X is compact.
Then

$$f_{x_1}+\cdots\cdots+f_{x_n}\in C(X),$$

and for each point $x\in X$ there exists an integer $j\,(1\leq j\leq n)$ such that $f_{x_j}(x)\neq 0$. Put

$$g=f_{x_1}\cdot\bar{f}_{x_1}+\cdots\cdots+f_{x_n}\cdot\bar{f}_{x_n}$$

then $g\in\mathrm{Ker}\ h$ and $g(x)\neq 0$ for all $x\in X$. Hence the maximal ideal Ker h contains an invertible element g. This is a contradiction. Thus there exists an element $x\in X$ such that $h=h_x$. That is, we have proved that

$$\phi:\ X\longrightarrow\triangle\ \ (x\longmapsto h_x)$$

is bijective.

Let γ be the weak topology of X induced by $C(X)$. In the original topology τ of X, since each element of $C(X)$ is continuous, it follows that $\gamma\subset\tau$. Since γ is a Hausdorff topology and τ is a compact topology, we have

$$\gamma=\tau\ \ ([1],\ [12]).$$

In the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ \phi\ } & \triangle & \xrightarrow{\ \hat{f}\ } & \boldsymbol{C} \\ \Cup & & \Cup & & \Cup \\ x & \longmapsto & h_x & \longmapsto & f(x) \end{array}$$

$(f \in C(X))$, take an open neighborhood U_x of $f(x)$ in $\boldsymbol{C}$, then it is clear that

$$\psi^{-1}(\hat{f}^{-1}(U_x)) = f^{-1}(U_x),$$

which is an open neighborhood of x in X (note that $\hat{f}$ is continuous). It is evident that ψ is a continuous open mapping, and thus

$$\psi: X \approx \triangle \text{ (homeomorphic).}$$

(note that the Gelfand topology of $\triangle$ is equivalent to γ and γ is equivalent to τ.) We can put $X = \triangle$.

In view of this point we can regard as

$$\begin{array}{rccc} f = \hat{f}: & X & \longrightarrow & \boldsymbol{C} \\ & \Psi & & \Psi \\ & x & \longmapsto & f(x) \end{array}$$

for each $f \in C(X)$ $(\hat{f}(h_x) = f(x))$. That is, we have the one-to-one corresponding:

$$\begin{array}{ccc} C(X) & \longleftrightarrow & C(\triangle) \\ \Psi & & \Psi \\ f & \longleftrightarrow & \hat{f}, \end{array}$$

and hence the commutative Banach algebra $C(X)$ is a semisimple algebra. ///

Corollary 3.2 Under the circumstance of Theorem 3.1, there is an one-to-one correspondence between the set of all closed subsets of X and the set of all closed ideals of $C(X)$ if each closed ideal of $C(X)$ is an intersection of maximal ideals.

Proof. Take a closed subset K of X and put

$$I_K = \bigcap_{x \in K} \operatorname{Ker} h_x,$$

where $h_x \in \triangle$ is defined by the same way as in the proof of theorem 3.1. Since each Ker h_x is a closed maximal ideal of $C(X)$, it follows that I_K is a closed ideal of $C(X)$.
Now let $C_s(X)$ be the set of all closed subsets of X and $C_I(C(X))$ a collection of all closed ideals of $C(X)$ which are an intersection of maximal ideals.
Define

$$\begin{array}{rccc} \psi: & C_s(X) & \longrightarrow & C_I(C(X)) \\ & \Psi & & \Psi \\ \text{by} & K & \longmapsto & I_K \end{array} .$$

If K and K' are closed subsets of X which is $K \neq K'$, then there is a point y such that

either $y \in K'$ and $y \notin K$ or $y \notin K'$ and $y \in K$.
Therefore, since X is normal, there is a continuous map $f: X \longrightarrow [0,1]$ such that $f(K) = 0, f(y) = 1$. Hence, since $I_K \neq I_{K'}$, ψ is one-to-one.
The inverse Φ of ψ is defined as follows. Let I be a closed ideal of $C(X)$. Put

$$K_I = \bigcap_{f \in I} f^{-1}(0). \quad \cdots\cdots\cdots\cdots (3\text{—}1)$$

Then K_I is a closed subset of X, because f is continuous, i.e., $f^{-1}(0)$ is closed in X.
Define

$$\Phi(I) = K_I$$

for each closed ideal I of $C(X)$. In this case, for each $x \in K_I$ it follows that

$$\forall f \in I, \; f(x) = 0,$$

which implies that $I \subset \text{Ker } h_x$. Therefore we have

$$I \subset \bigcap_{x \in K_I} \text{Ker } h_x.$$

By our hypothesis there are maximal ideals $M_\alpha(\alpha \in \Lambda$, Λ is an indexing set) of $C(X)$ such that

$$I = \bigcap_{\alpha \in \Lambda} M_\alpha.$$

Hence

$$\bigcap_{\alpha \in \Lambda} M_\alpha \subset \bigcap_{x \in K_I} \text{Ker } h_x.$$

Assume that there is an element $\alpha \in \Lambda$ such that

$$M_\alpha \notin \{\text{Ker } h_x \mid x \in K_I\}.$$

Then there exists an element $y \in X$ with $\text{Ker } h_y = M_\alpha$ by theorem 3.1.
Since $y \notin K_I$, we have

$$\bigcap_{f \in I} f^{-1}(0) \supset K_I \cup \{y\},$$

which is a contradiction to (3—1). Therefore it follows that

$$I = \bigcap_{x \in K_I} \text{Ker } h_x.$$

(note that each maximal ideal is a kernel of an element of $\triangle$ and the inverse is also true.) That is, $\psi\Phi$ is the identity map of $C_I(C(X))$ onto itself. ψ is onto and thus ψ is bijective. ///

Example 3.3 When $X=\{a_1, a_2, a_3\}$ has a discrete topology, it follows that $C(X)=C^3$ which is a commutative Banach algebra with norm $\|x\|=|x|$ for each $x\in C^3$. Then, by theorem 3.1, the maximal ideal space $\triangle$ of $C(X)$ is homeomorphic to $X=\{a_1, a_2, a_3\}$, i.e., $\triangle=\{h_{a_1}, h_{a_2}, h_{a_3}\}$, where for each $x=(x_1, x_2, x_3)\in C^3$

$$h_{a_i}(x)=x(a_i)=x_i \quad (i=1,2,3).$$

Note that the addition, multiplication and scalar product are defined by

$$x+y=(x_1+y_1,\ x_2+y_2,\ x_3+y_3),$$
$$xy=(x_1y_1,\ x_2y_2,\ x_3y_3),$$
$$\lambda x=(\lambda x_1,\ \lambda x_2,\ \lambda x_3),$$

where $x=(x_1, x_2, x_3)$, $y=(y_1, y_2, y_3)\in C^3$ and $\lambda\in C$. In this case:

the set of closed subsets of $X=\{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1,a_2\}, \{a_1,a_3\}, \{a_2,a_3\}, \{a_1,a_2,a_3\}\}$ on the other hand,

$$\text{the set of maximal ideals}=\begin{Bmatrix}\text{Ker } h_{a_1}=M_1=(x_1,\ x_2,\ x_3)\in C^3\,|\,x_1=0\\ \text{Ker } h_{a_2}=M_2=(x_1,\ x_2,\ x_3)\in C^3\,|\,x_2=0\\ \text{Ker } h_{a_3}=M_3=(x_1,\ x_2,\ x_3)\in C^3\,|\,x_3=0\end{Bmatrix}$$

$$\text{the set of closed ideals (not maximal)}=\begin{Bmatrix}A_0=\bigcap_{\alpha\in\phi} M_\alpha=C^3\\ A_1=\{(x_1,x_2,x_3)\in C^3\,|\,x_2=x_3=0\}=M_2\cap M_3\\ A_2=\{(x_1,x_2,x_3)\in C^3\,|\,x_1=x_3=0\}=M_1\cap M_3\\ A_3=\{(x_1,x_2,x_3)\in C^3\,|\,x_1=x_2=0\}=M_1\cap M_2\\ A_4=\{(0,0,0)\in C^3\}=M_1\cap M_2\cap M_3\end{Bmatrix}$$

Thus, each closed ideal is an intersection of maximal ideals. Our example satisfies the hypothesis of corollary 3.2, and we have the one-to-one corresponding as follows:

The set of closed subsets		The set of closed ideals
ϕ	$\longleftrightarrow$	C^3
$\{a_1\}$	$\longleftrightarrow$	$\text{Ker } h_{a_1}=M_1$
$\{a_2\}$	$\longleftrightarrow$	$\text{Ker } h_{a_2}=M_2$
$\{a_3\}$	$\longleftrightarrow$	$\text{Ker } h_{a_3}=M_3$
$\{a_1,a_2\}$	$\longleftrightarrow$	$M_1\cap M_2=A_3$
$\{a_1,a_3\}$	$\longleftrightarrow$	$M_1\cap M_3=A_2$
$\{a_2,a_3\}$	$\longleftrightarrow$	$M_2\cap M_3=A_1$
$\{a_1,a_2,a_3\}$	$\longleftrightarrow$	$M_1\cap M_2\cap M_3=A_4$

Proposition 3.4 Let A and B be commutative Banach algebras and let B be semisimple. For a homomorphism $\psi: A \longrightarrow B$ such that $\psi(A)$ is dense in B, assume that $\alpha: \triangle_B \longrightarrow \triangle_A$ is defined by

$$(\alpha h)(x) = h(\psi(x)) \quad (x \in A, \ h \in \triangle_B),$$

where $\triangle_A$ and $\triangle_B$ are maximal ideal spaces of A and B, respectively. Then $\alpha(\triangle_B)$ is a compact subset of $\triangle_A$ and $\alpha: \triangle_B \longrightarrow \alpha(\triangle_B) \subset \triangle_A$ is injective.

Proof. By Lemma 2.5, ψ is a continuous map. In the diagram

$$\begin{array}{ccccc} \triangle_B & \xrightarrow{\ \alpha\ } & \triangle_A & \xrightarrow{\ \hat{x}\ } & C \\ \Cup & & \Cup & & \Cup \\ h & \longmapsto & \alpha h & \longmapsto & h(\psi(x)) \quad (x \in A) \end{array}$$

we see that

(a) $h \circ \psi: A \xrightarrow{\psi} B \xrightarrow{h} C$ is a complex homomorphism (continuous), and thus $\alpha: \triangle_B \longrightarrow \triangle_A$ is well-defined,

(b) for an open neighborhood U of $h(\psi(x))$ in C, $\hat{x}^{-1}(U)$ is an open neighborhood of αh in $\triangle_A$, and furthermore

$$\alpha^{-1}(\hat{x}^{-1}(U)) = \widehat{\psi(x)}^{-1}(U).$$

Noting that the topology of B is completely determined by $\psi(A)$ because $\psi(A)$ is dense in B, it is clear that α is continuous, and thus $\alpha(\triangle_B)$ is a compact subset of $\triangle_A$($\triangle_B$ is compact).

For $h_1 \neq h_2$ in $\triangle_B$, there exists at least one point $y \in B$ such that $h_1(y) \neq h_2(y)$. Since $\psi(A)$ is dense in B, there is a sequence $\{x_i\} \subset A$ such that $\psi(x_n) \longrightarrow y$. By the continuities of h_1 and h_2, we have

$$h_1(\psi(x_n)) \longrightarrow h_1(y), \quad h_2(\psi(x_n)) \longrightarrow h_2(y).$$

The fact that $h_1(y) \neq h_2(y)$ implies that there exists $x_n \in A$ such that $h_1(\psi(x_n)) \neq h_2(\psi(x_n))$. Therefore

$$\alpha h_1 \neq \alpha h_2,$$

that is, α is injective. ///

4. B*-Algebras

We put

$$D=\{z\in C \mid |z|\leq 1\}$$

and denote the set of all holomorphic functions from D to C by $H(D)$. Then, by the usual operations $H(D)$ is an algebra over C. Define a norm and an involution on $H(D)$ by setting

(i) $\|f\|=\sup_{z\in D}|f(z)|,\ f\in H(D)$

(ii) $f^*(z)=\overline{f(\bar{z})},\ z\in D,\ f\in H(D).$

Then, $H(D)$ is a commutative Banach algebra with an involution $*$(see Definition 2.4), which is called the *disc algebra.*

In this section, we shall prove some properties on B^*-algebras and $H(D)$, and illustrate a counterexample about Lemma 2.6.

Theorem 4.1 For a commutative B^*-algebra A, the following hold.

(i) A is semisimple.

(ii) If $x\in A$ is hermitian such that $\|x\|=\alpha$ and $x\geq 0$, then $\sigma(\alpha e-x)\subset[0,\alpha]$ and $\sigma(x)\subset[0,\alpha]$.

(iii) If $x\in A$ is hermitian, then $x^2\geq 0$.

Proof. (i) For each $x\in A$,

$$\|x\|^2=\|xx^*\|\leq\|x\|\ \|x^*\|\Longrightarrow\|x\|\leq\|x^*\|$$
$$\Longrightarrow\|x^*\|\leq\|x^{**}\|=\|x\|$$

and thus

$$\|x\|=\|x^*\|,\ \|x\|^2=\|x\|\ \|x^*\| \quad\cdots\cdots\cdots(4-1)$$

Put $y=xx^*$. Then y is hermitian. Thus

$$\|y\|^2=\|yy^*\|=\|y^2\|.$$

By induction on n, it follows that for $m=2^n$, $\|y^m\|=\|y\|^m$ ($n=2\Rightarrow\|y^4\|=\|(y^2)^2\|=\|y^2\|^2=(\|y\|^2)^2=\|y\|^4$). By 2° and 4° in §2,

$$\|\hat{y}\|_\infty=\rho(y)=\lim_{m\to\infty}\|y^m\|^{1/m}=\|y\|.$$

For each $h\in\triangle$,

$$\hat{y}(h)=\hat{x}(h)\hat{x}^*(h)=|\hat{x}(h)|^2 \text{ (by 5° in §2)},$$

i.e., $\hat{y}=|\hat{x}|^2$, and thus

$$\|\hat{x}\|^2_\infty=\|\hat{y}\|_\infty=\|y\|=\|xx^*\|=\|x\|^2$$

(by (4—1)). In consequence, for each $x\in A$ we have

$$\|\hat{x}\|_\infty=\|x\|.$$

Take $x\in Rd(A)$. Then

$$\begin{aligned}\forall h\in\triangle,\ \hat{x}(h)=h(x)=0 &\Longrightarrow \|\hat{x}\|_\infty=0\\ &\Longrightarrow \|x\|=0\\ &\Longrightarrow x=0,\end{aligned}$$

where $\triangle$ is the maximal ideal space of A, and hence $Rd(A)=\{0\}$.

(ii) Let $x\in A$ be hermitian such that $\sigma(x)\subset[0,\infty)$ and $\|x\|=\alpha$. If $|\beta|>\alpha$ ($\beta\in \boldsymbol{R}$), then

$$(\beta e-x)=\beta(e-x/\beta)$$

is invertible because that $\|x/\beta\|<1$ ([13], [14], [15]).

Therefore, $\sigma(x)\subset[0,\alpha]$ is clear. Noting that $\alpha e-x$ is also hermitian it follows from 4° and 5° in §2 that $\sigma(\alpha e-x)\subset\boldsymbol{R}$.

$\lambda>\alpha\Longrightarrow\lambda e-(\alpha e-x)=-((\alpha-\lambda)e-x)$ is invertible because $\alpha-\lambda<0$ and $\sigma(x)\subset[0,\infty)$,

$\lambda<0\Longrightarrow\lambda e-(\alpha e-x)=-((\alpha-\lambda)e-x)$ is invertible because $\alpha-\lambda>\alpha$ and $\sigma(x)\subset[0,\alpha]$.

Therefore $\sigma(\alpha e-x)\subset[0,\alpha]$.

(iii) It suffices to prove that $\sigma(x^2)\subset[0,\infty)$ because

$$(x^2)^*=(xx)^*=x^*x^*=x\cdot x=x^2$$

for each hermitian element $x\in A$. Put

$$\Omega=\text{an open neighborhood of } \boldsymbol{R} \text{ in } \boldsymbol{C}$$

and define

$$f:\ \Omega\longrightarrow\boldsymbol{C}\ \ (\lambda\longmapsto f(\lambda)=\lambda,\lambda\in\Omega)$$

(note that since x is hermitian $\sigma(x)\subset\boldsymbol{R}$ by 5° in §2). Since f is a holomorphic function, by Lemma 2.3

$$\tilde{f}(x)=x=\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\,(\lambda e-x)^{-1}\,d\lambda,$$

where Γ is any contour that surrounds $\sigma(x)$ in Ω (since $\hat{x}$ is continuous, $\{\hat{x}(h) \mid h\in\triangle\}$ $=\sigma(x)$ is bounded in $\boldsymbol{R}$. That is, the existence of Γ is obvious). Moreover, since

$$x^2=\tilde{f}(x)\cdot\tilde{f}(x)=\frac{1}{2\pi i}\int_{\Gamma} f^2(\lambda)\,(\lambda e-x)^{-1}\,d\lambda$$

(see Lemma 2.3), by the spectral mapping theorem (3° in §2)

$$\sigma(x^2)=f^2\ (\sigma(x))\subset[0,\infty). \qquad ///$$

Proposition 4.2 Let $H(D)$ be the disc algebra.

(i) $H(D)$ is a commutative B^*-algebra.

(ii) For every $f\in H(D)$, $ff^*\geq 0$.

(iii) The mapping

$$F:\ H(D)\longrightarrow \boldsymbol{C}$$

defined by

$$F(f)=\int_{-1}^{1} f(t)\,dt \quad (f\in H(D))$$

is a positive functional.

Proof. (i) By the Runge's theorem, there are rational functions $\{f_n\}$ which approximate a holomorphic function $f\in H(D)$. Therefore we have

$$\overline{f_n(\bar{z})}\longrightarrow f^*(z)=\overline{f(\bar{z})}.$$

Since $f_n\ (z)$ is a rational function $f_n(\bar{z})=\overline{f_n(z)}$, and thus $\overline{f_n(\bar{z})}=f_n(z)$. In the diagram

$$\begin{array}{ccc} |\overline{f_n(\bar{z})}| & \longrightarrow & |f^*(z)|=|\overline{f(\bar{z})}| \\ \| & & \\ |f_n(z)| & \longrightarrow & |f(z)|, \end{array}$$

we get $|\overline{f(\bar{z})}|=|f^*(z)|=|f(z)|$. Thus,

$$\begin{aligned}\|ff^*\|&=\sup_{z\in D}|(ff^*)\ (z)|=\sup_{z\in D}|f(z)\cdot f^*(z)|\\ &=\sup_{z\in D}|f(z)|\ |f^*(z)|\end{aligned}$$

$$=\sup_{z\in D}|f(z)|^2$$
$$=\|f\|^2.$$

That is, $H(D)$ is a B^*-algebra.

(ii) By (i) of Theorem 4,1, $H(D)$ is semisimple, and thus $H(D)=\widehat{H(D)}=C(\triangle)$ ([5], [6], [12]), where $\triangle$ is the maximal ideal space of $H(D)$ and

$$C(\triangle)=\{f:\ \triangle\longrightarrow \boldsymbol{C}\,|\,f \text{ is continuous}\}.$$

Put $g=ff^*$. Then g is is hermitian. By $4°$ and $5°$ in §2,

$$\sigma(g)=\hat{g}(\triangle)\subset \boldsymbol{R}.$$

and thus we shall prove that $\hat{g}(\triangle)\subset[0,\infty)$. Since $\widehat{H(D)}=C(\triangle)$ and $|\hat{g}|-\hat{g}\in C(\triangle)$ $(\forall h\in\triangle, |\hat{g}|(h)=|\hat{g}(h)|)$, there exists an element $z\in H(D)$ such that

$$\hat{z}=|\hat{g}|-\hat{g}\quad (\text{on } \triangle). \qquad \cdots\cdots\cdots(4\text{—}2)$$

Then $\hat{z}(\triangle)=|\hat{g}(\triangle)|-\hat{g}(\triangle)\subset\boldsymbol{R}$, which implies that $z\in H(D)$ is hermitian (see $5°$ in §2). Put

$$zf=w=u+iv,$$

where u and v are hermitian elements of $H(D)$ (for the existence of u and v see [1] or [12]). Then

$$ww^*=zf(zf)^*=z^2\cdot g,$$
$$w^*w+ww^*=2u^2+2v^2$$
$$w^*w=2u^2+2v^2-ww^*=2u^2+2v^2-z^2g \qquad \cdots\cdots\cdots(4\text{—}3)$$

By (iii) of Theorem 4.1, $u^2\geq 0$ and $v^2\geq 0$. It is easy to see that

$$(\hat{z}^2\cdot\hat{g})(h)=((|\hat{g}|-\hat{g})\cdot(|\hat{g}|-\hat{g})\cdot\hat{g})(h)\leq 0$$

for all $h\in\triangle$. Therefore, $-\hat{z}^2\cdot\hat{g}(\triangle)\subset[0,\infty)$ and $-\hat{z}\cdot g\geq 0$ ($z^2\cdot g$ is hermitian).
By Lemma 2.6 and (4—3), $w^*w\geq 0$.
Since $H(D)$ is commutative and $w^*w\geq 0$, $ww^*\geq 0$,

$$ww^*=z^2g\geq 0, \text{ and therefore in (4—2)},$$

we have to have $|\hat{g}|=\hat{g}$. This implies that $g=ff^*\geq 0$.

(iii) For each $f \in H(D)$,

$$F(ff^*) = \int_{-1}^{1} (ff^*)(t)dt = \int_{-1}^{1} f(t)\overline{f(\bar{t})}\, dt$$
$$= \int_{-1}^{1} f(t) \cdot \overline{f(t)} dt = \int_{-1}^{1} |f(t)|^2 dt \geq 0,$$

and F is linear. ///

We have to note that Lemma 2.6 insists that if $xy = yx$ then $\sigma(x+y) \subset \sigma(x) + \sigma(y)$ and $\sigma(xy) \subset \sigma(x) \cdot \sigma(y)$. When $xy \neq yx$, we have a counterexample about this Lemma as follows.

Example 4.3 We define a norm on C^2 by

$$\|w\| = |w_1| + t|w_2|$$

for a large positive number t, where $w = (w_1, w_2) \in C^2$.
Let A be the algebra of all complex 2-by-2 matrices with the usual operations. Define a norm on A by

$$\|y\| = \max\{\|y(w)\| \mid w \in C^2, \|w\| = 1\}$$

for $y \in A$. Then A is a Banach algebra (non-commutative).

Proof. Suppose

$$\left\{ y_n = \begin{pmatrix} y_1(n) & y_2(n) \\ y_3(n) & y_4(n) \end{pmatrix} \right\}$$

is a Cauchy sequence in A. Then, for each positive number ε, there exists a positive integer N such that

$$\forall n, m \geq N \Longrightarrow \|y_n - y_m\| < \varepsilon.$$

Since

$$y_n - y_m = \begin{pmatrix} y_1(n) - y_1(m) & y_2(n) - y_2(m) \\ y_3(n) - y_3(m) & y_4(n) - y_4(m) \end{pmatrix},$$

$\|y_n - y_m\| < \varepsilon$ implies that $|y_i(n) - y_i(m)|$ $(i = 1, 2, 3, 4)$ is sufficiently small. That is,

$$\{y_n\} \text{ is a Cauchy sequence} \Longrightarrow \{y_i(n)\}\ (i = 1, 2, 3, 4) \text{ is a Cauchy sequence.}$$

Thus, $y_i(n) \longrightarrow x_i (\in \mathbf{C})$ and thus

$$y_n = \begin{pmatrix} y_1(n) & y_2(n) \\ y_3(n) & y_4(n) \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

That is, A is a Banach space.

Consider $w=(w_1, w_2)$ with $\|w\|=1$ and

$$x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad y=\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix},$$

where $w\in\mathbf{C}^2$, $x, y\in A$. Then

$$\begin{aligned} xyw &= \begin{pmatrix} x_1y_1+x_2y_3 & x_1y_2+x_2y_4 \\ x_3y_1+x_4y_3 & x_3y_2+x_4y_4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} (x_1y_1+x_2y_3)\, w_1+(x_1y_2+x_2y_4)\, w_2 \\ (x_3y_1+x_4y_3)\, w_1+(x_3y_2+x_4y_4)\, w_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \|xyw\| &= |(x_1y_1+x_2y_3)\, w_1+(x_1y_2+x_2y_4)\, w_2|+t|(x_3y_1+x_4y_3)\, w_1+(x_3y_2+x_4y_4)w_2| \\ &= |x_1(y_1w_1+y_2w_2)+x_2(y_3w_1+y_4w_2)|+t|x_3(y_1w_1+y_2w_2)+x_4(y_3w_1+y_4w_2)| \\ &\leq |y_1w_1+y_2w_2|(|x_1|+t|x_3|)+|y_3w_1+y_4w_2|(|x_2|+t|x_4|). \quad \cdots\cdots(4\text{—}4) \end{aligned}$$

Note that

$$\|(w_1, w_2)\|=1 \Longrightarrow |y_1w_1+y_2w_2|+t|y_3w_1+y_4w_2|\leq\|y\|$$

$$\|(1, 0)\|=1=\|(0, \tfrac{1}{t})\| \Longrightarrow |x_1|+t|x_3|\leq\|x\|, \ \frac{|x_2|}{t}+|x_4|\leq\|x\|$$

Therefore, by (4—4)

$$\|xyw\|\leq\|x\|\cdot\|y\| \Longrightarrow \|xy\|\leq\|x\|\cdot\|y\|.$$

For the identity $e=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of A and $w\in\mathbf{C}^2$ with $\|w\|=1$,

$$1=\|w\|=\|ew\|,$$

and thus $\|e\|=1$. Therefore A is a Banach algebra. ///

We define $*$: $A\longrightarrow A$ by

$$x^*=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^*=\begin{pmatrix} \bar{x}_1 & \bar{x}_3 \\ \bar{x}_2 & \bar{x}_4 \end{pmatrix}.$$

Then it is easy to prove that $*$ is an involution. Therefore A is a Banach algebra with involution $*$.

Consider a fixed element $x\in A$ such that

$$x=\begin{pmatrix} 0 & t^2 \\ 1 & 0 \end{pmatrix}.$$

Then the following hold.

(i) $\|x(w)\|=t\|w\|$ for each $w\in\mathbf{C}^2$, i.e., $\|x\|=t$

(ii) $\sigma(x)=\{t, -t\}=\sigma(x^*)$

(iii) $\sigma(xx^*)=\{1, t^4\}=\sigma(x^*x)$, $\sigma(x+x^*)=\{1+t^2, -1-t^2\}$.

Proof (i) For each $w=(w_1, w_2)$,

$$x(w)=\begin{pmatrix}0 & t^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}t^2\cdot w_2\\ w_1\end{pmatrix}$$

and hence $\|x(w)\|=t^2|w_2|+t|w_1|=t(|w_1|+t|w_2|)=t\|w\|$.

(ii) Consider

$$\lambda e-x=\begin{pmatrix}\lambda & -t^2\\ -1 & \lambda\end{pmatrix}.$$

$\lambda e-x$ is not invertible $\Longleftrightarrow \lambda^2-t^2=0 \Longleftrightarrow \lambda=\pm t$. Hence

$$\sigma(x)=\{+t, -t\}.$$

Since

$$x^*=\begin{pmatrix}0 & 1\\ t^2 & 0\end{pmatrix},$$

we have also $\sigma(x^*)=\{+t, -t\}$.

(iii) Since

$$x+x^*=\begin{pmatrix}0 & 1+t^2\\ 1+t^2 & 0\end{pmatrix}, \quad xx^*=\begin{pmatrix}t^4 & 0\\ 0 & 1\end{pmatrix},$$

It follows that

$$\sigma(x+x^*)=\{\pm(1+t^2)\}, \quad \sigma(xx^*)=\{1, t^4\}=\sigma(x^*x). \qquad ///$$

In this case, $xx^*\neq x^*x$ and

$$\sigma(x+x^*)=\{\pm(1+t^2)\}\not\subset\sigma(x)+\sigma(x^*)=\{-2t, 0, 2t\},$$
$$\sigma(xx^*)=\{1, t^4\}\not\subset\sigma(x)\cdot\sigma(x^*)=\{-t^2, t^2\}.$$

That is, Lemma 2.6 does not hold.

References

1. S.K. Berberian: Lectures in Functional Analysis and Operator Theory, Springer-Verlag New York Inc. (1974).
2. F.F. Bonsall and J.Duncan: Numerical ranges of operators on normed spaces and of elements of normed algebras, London Math. Soc. Lecture Note Series No. 2 Cambridge: Cambridge Univ. Press (1971).
3. N.Bourbaki: Théories spectrales. Chs. I. II. Fasc. XXXII of Éléments de mathématique. Paris, Hermann (1967).
4. J.Y.Choi: Integrals and differentiations on Banach algebras and their applications (to appear).
5. R.G.Douglas and C.Pearcy: On the Spectral theorem for normal Operators, *Proc. Cambridge Philos. Soc.* ***68***, pp.393—400 (1970).
6. P.R.Halmos: A Hilbert space problem book, Princeton, N.J., Van Nostrand (1967).
7. P.R.Halmos: Continuous functions of Hermitian operators. *Proc. Amer. Math. Soc.* ***31***, pp.130—132 (1972).
8. S.Kantorovitz: Classification of operators by means of their operational calculus; *Trans. Amer. Math. Soc.* ***115***, pp.194—224 (1965).
9. M.Krein: On Positive and additive functionals in linear normed spaces, Comm. Inst. Sci. Math. Méc. Univ. Kharkaff (4)14, pp.227-237 (1937).
10. M.Krein and D.Milman: On extreme points of regularly convex sets, *Studia Math.* ***9***, pp.133—138 (1940).
11. C.E.Rickart: General theory of Banach algebras, Princeton, N.J., Van Nostrand (1960).
12. W.Rudin: Functional Analysis, McGraw-Hill, Inc. (1973).
13. S.Sakai: C^*-algebras and W^*-algebras, Springer-Verlag (1971).
14. J.Wermer: Banach algebras and analytic functionals, *Adv. Math.* ***1***, pp.51—102 (1961).
15. K.Yosida: Functional Analysis, Springer-Verlag (1965).

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