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Elliptic Operators on Compact Differentiable Manifolds

by

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1. Introduction

It can be observed in the Harmonic Integral Theory that there is an interdisciplinarity between Analysis and Topology. One of the famous theorems in this field is the Hodge Theorem, which was completed approximately 1950. Today, it appears that this Theorem is utilized in each field of Topology and Analysis ([1], [2], [7], [9]). In particular, it is well known that it led to the Atyah's Index Theorem ([6]).

In this paper, we investigate some properties of Laplacian operator $\triangle$ which is indispensable in Hodge Theorem (§4), and discuss on eigenvalues of $\triangle$. We also deal with the generalization of the Hodge Theorem. It can, in more details, be described as follows.

Section 2 presents the general theory of Differential operator and Symbol which is a radical background for the contents of section 3 and 4. We also prove Proposition 2.4 and Lemma 2.6 as the preliminaries for section 3 and 4.

In §3, using elliptic complex, we define Laplacian operator $\triangle$ more generally, and prove that if

$$E:\ 0\longrightarrow\mathscr{E}(M,E_0)\longrightarrow\cdots\longrightarrow\mathscr{E}(M,E_N)\longrightarrow 0$$

is an elliptic complex, then

$$H^q(E)\cong\ker\ \triangle.$$

In §4, we investigate some properties on eigenvalue of $\triangle$ in Proposition 4.6, and prove that there is a sequence of eigenvalues

$$0\leq\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots\cdots$$

of $\triangle$ in Theorem 4.7, which is the main theorem of this paper.

2. Differential Operators

In this and next sections, by M we mean a compact differentiable manifold with $\dim_R M=n$, and by a vector bundle we mean a finite dimensional $\boldsymbol{C}$-vector bundle over M, where $\boldsymbol{R}$ is the set of all reals and $\boldsymbol{C}$ is the set of all complexes.

For a vector bundle $E\longrightarrow M$ there is a Hermitian metric $<\ ,\ >_E$ on E ([6], [12]), and we put

$$\mathscr{E}(M,E)=\text{the set of all } C^\infty\text{-sections of } E,$$

where a C^∞-section is a section which is of the C^∞-class.

Define an inner product $(\ ,\ )$ on $\mathscr{E}(M,E)$ by

$$(\xi,\eta)=\int_M <\xi(x),\ \eta(x)>_E\, dx \qquad \cdots\cdots(2\text{—}1)$$

for $\xi,\eta\in\mathscr{E}(M,E)$.

Let $\|\xi\|_0=(\xi,\xi)^{1/2}$ $(\xi\in\mathscr{E}(M,E))$ be the L^2-norm on $\mathscr{E}(M,E)$ and let $W^0(M,E)$ be the completion of $\mathscr{E}(M,E)$ with respect to the L^2-norm.

We assume that $\{U_\alpha,\ \varphi_\alpha\}$ is a finite trivializing covering of M, then for each α there is the commutative diagram

$$\begin{array}{ccc} E|U_\alpha & \xrightarrow{\ \ \varphi_\alpha\ \ } & \tilde U_\alpha\times \boldsymbol{C}^m \\ \downarrow & & \downarrow \\ U_\alpha & \xrightarrow{\ \ \tilde\varphi_\alpha\ \ } & \tilde U_\alpha\ \text{(open in } R^n), \end{array}$$

where φ_α is a bundle isomorphism and $\tilde\varphi_\alpha$: $U_\alpha\longrightarrow\tilde U_\alpha$ are local coordinate systems for M. Hence if we put

$$\mathscr{E}(\tilde U_\alpha)=\{f:\ \tilde U_\alpha\longrightarrow \mathbf{C}\,|\,f \text{ is of the } C^\infty\text{-class}\},$$

then there is the induced map

$$\varphi_\alpha^*:\ \mathscr{E}(U_\alpha,E)\longrightarrow[\mathscr{E}(\tilde U_\alpha)]^m.$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and define, for $\xi\in\mathscr{E}(X,E)$,

$$\|\xi\|_{s,E}=\sum_\alpha\|\varphi_\alpha^*\,\rho_\alpha\,\xi\|_{s,R^n},$$

where $\|\ \|_{s,R^n}$ is the Sobolev norm which is defined as follows.

Let f: $\boldsymbol{R}^n\longrightarrow\boldsymbol{C}^m$ be a compactly supported differentiable function. Then we have

$$\|f\|^2_{s,R^n}=\int_{R^n}|\hat{f}(y)|^2(1+|y|^2)^s\,dy,$$

where $\hat{f}$ is the Fourier transformation of f, *i.e.*,

$$\hat{f}(y)=\frac{1}{(2\pi)^n}\int_{R^n} f(x)\,e^{-ix\cdot y}\,dx$$

and for $y=(y_1,\cdots,y_n)$ $|y|=(y_1^2+\cdots+y_n^2)^{1/2}$.

Then $\|f\|_{o,R^n}=\|\hat{f}\|_o$.

Definition 2.1. The $W^s(M,E)$ is defined to be the completion of $\mathscr{E}(M,E)$ with respect to the Sobolev norm $\|\ \|_{s,E}$.

It is easy to prove that

1^o. $W^s(M,E)$ is a Hilbert space and

$$\cdots\supset W^s(M,E)\supset W^{s+1}(M,E)\supset\cdots$$

2^o. if $(W^s(M,E))^*=\{f:\ W^s(M,E)\longrightarrow \mathbf{C}\,|\,f:$ is a conjugate linear continuous functionals$\}$,

then $(W^s(M,E))^*\cong W^{-s}(M,E)$.

Definition 2.2. Let $E\longrightarrow M$ and $F\longrightarrow M$ be $\mathbf{C}$-vector bundles and let $L:\ \mathscr{E}(M,E)\longrightarrow\mathscr{E}(M,F)$ be a $\mathbf{C}$-linear map.

If for any choice of local coordinates and local trivializing cover of M there exists a linear partial differential operator $\tilde{L}$ such that the diagram

$$\begin{array}{ccc}
[\mathscr{E}(U)]^p & \xrightarrow{\quad\tilde{L}\quad} & [\mathscr{E}(U)]^q \\
\wr\| & & \wr\| \\
\mathscr{E}(U,U\times\mathbf{C}^p) & \longrightarrow & \mathscr{E}(U,U\times\mathbf{C}^q) \\
\cup & & \cup \\
\mathscr{E}(M,E)|_U & \xrightarrow{\quad L\quad} & \mathscr{E}(M,F)|_U
\end{array}$$

is commutative, where $\dim_C E=p$ and $\dim_C F=q$, then L is called a *differential operator*.

If for each $f=(f_1,\cdots,f_p)\in[\mathscr{E}(U)]^p$

$$\tilde{L}(f)=\sum_{|\alpha|\le k}\begin{pmatrix} a^{11}D^\alpha \cdots\cdots\cdots a^{1p}D^\alpha \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a^{q1}D^\alpha\cdots\cdots\cdots a^{qp}D^\alpha\end{pmatrix}\begin{pmatrix} f_1\\ \vdots \\ f_p\end{pmatrix}$$

then L is said to be of *order* k, where for $\alpha=(\alpha_1,\cdots,\alpha_n)$ (each α_i is a nonnegative integer)

$$|\alpha|=\alpha_1+\cdots\cdots+\alpha_n$$

and

$$D^{\alpha}=(-i)^{|\alpha|}\, D_1^{\alpha_1}\cdots\cdots D_n^{\alpha_n},\ D_j=\frac{\partial}{\partial x_j}\ (j=1,\cdots,n).$$

We put

$$\mathrm{Diff}_k(E,F)=\{L:\ \mathscr{E}(M,E)\longrightarrow\mathscr{E}(M,F)\,|\,L \text{ is a differential operator of order } k\}.$$

Definition 2.3. Under the above notation, let

$$T:\ \mathscr{E}(M,E)\longrightarrow\mathscr{E}(M,F)$$

be a $\boldsymbol{C}$-linear map. If there is a continuous extension

$$T_s:\ W^s(M,E)\longrightarrow W^{s-k}(M,F)$$

of T for each $s\in\boldsymbol{R}$, then T is called an *operator of order k*. We put

$$OP_k(E,F)=\{L:\ \mathscr{E}(M,E)\longrightarrow\mathscr{E}(M,F)\,|\,L \text{ is an operator of order } k\}.$$

Proposition 2.4. (i) $\mathrm{Diff}_k(E,F)\subset OP_k(E,F)$

(ii) For each $L\in OP_k(E,F)$ there exists the adjoint operator $L^*\in OP_k(F,E)$ of L, and the extension

$$(L^*)_s:\ W^s(M,F)\longrightarrow W^{s-k}(M,E)$$

is given by the adjoint map

$$(L_{k-s})^*:\ W^s(M,F)\longrightarrow W^{s-k}(M,E).$$

Proof. (i) For any $f\in\mathscr{E}(M,E)$ with $\|f\|_{s+k,E}<\infty$, we must prove that $\|Lf\|_{s,E}<\infty$ for each $L\in\mathrm{Diff}_k(E,F)$.

Take a multi-index α with $|\alpha|=k$, then

$$\begin{aligned}\|D^{\alpha}f\|^2_{s,E}&=\int|\widehat{D^{\alpha}f}(y)|^2\ (1+|y|^2)^s\ dy\\&=\int y^{2\alpha}|\hat{f}(y)|^2\ (1+|y|^2)^s\ dy\\&\leq\int(1+|y|^2)^k\ |\hat{f}(y)|^2\ (1+|y|^2)^s\ dy\\&=\int|\hat{f}(y)|^2\ (1+|y|^2)^{s+k}\ dy\\&=\|f\|^2_{s+k,E}<\infty,\end{aligned}$$

where we used that $\widehat{D^\alpha f}(y)=y^\alpha \hat{f}(y)$ ([6], [8]) and $y^{2\alpha}=y_1^{2\alpha_1}\cdots y_n^{2\alpha_n}\leq(1+(y_1^2+\cdots y_n^2))^k=(1+|y|^2)^k$ for $y=(y_1,\cdots,y_n)$ and $\alpha=(\alpha_1,\cdots,\alpha_n)$.

By Definition 2,2, L is a polynomial in D with degree k, and thus

$$\|Lf\|_{s,E}<\infty.$$

(ii) By Definition 2.3, for each $L\in OP_k(E,F)$ and each $s\in\mathbf{R}$, there exists the continuous extension

$$\underline{L}_s:\ W^s(M,E)\longrightarrow W^{s-k}(M,F)$$

of L which is a linear map.

Since $W^s(M,E)$ and $W^{s-k}(M,F)$ are Hilbert spaces, there exists the adjoint operator

$$(\underline{L}_s)^*:\ (W^{s-k}(M,F))^*\longrightarrow(W^s(M,E))^*$$

of $\underline{L}_s$ for all $s\in\mathbf{R}$. Noting that

$$(W^s(M,E))^*\cong W^{-s}(M,E)\ (2^\circ \text{ above}),$$

we have the linear operator

$$(\underline{L}_{k-s})^*:\ W^s(M,F)\longrightarrow W^{s-k}(M,E).$$

We put $(\underline{L}^*)_s=(\underline{L}_{k-s})^*$, and define

$$L^*:\ \mathscr{E}(M,F)\longrightarrow\mathscr{E}(M,E)$$

by $L^*=(\underline{L}^*)_s|_{\mathscr{E}(M,E)}$, then L^* is the adjoint of L (Note that $\mathscr{E}(M,F)\subset W^s(M,F)$).

Definition 2.5. Let $T^*(M)$ be the cotangent bundle of M, and let $T'(M)=T^*(M)-M$. For the projection mapping $\pi:\ T'(M)\longrightarrow M$, π^*E and π^*F are pullbacks of vector bundles E and F over M, respectively.

We set, for each $k\in\mathbf{Z}$ (the set of all integers),

$$\begin{aligned}\mathrm{Smbl}_k(E,F)=\{\sigma\in\mathrm{Hom}(\pi^*E,\ \pi^*F)\,|\,\sigma(x,\ \rho v)=\rho^k\sigma(x,v),\\ (x,v)\in T'(M),\ \rho>0\},\end{aligned}$$

where $\mathrm{Hom}(\pi^*E,\ \pi^*F)$ is the set of all bundle homomorphisms from π^*E to π^*F. Each element of $\mathrm{Smbl}_k(E,F)$ is called a *k-symbol* from E to F.

For each differential operator $L\in\mathrm{Diff}_k(E,F)$ the k-symbol $\sigma_k(L)$ of L is defined as follows. Let $(x,v)\in T'(M)$ and $e\in E_x$ be given. Find $g\in\mathscr{E}(M)$ and $f\in\mathscr{E}(M,E)$ such that

$dg_x = v$, and $f(x) = e$.

We define

$$\sigma_k(L)(x, v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x).$$

Then it is easy to prove that $\sigma_k(L) \in \mathrm{Smbl}_k(E, F)$ ([6]). In our situation:

3° The sequence

$$0 \longrightarrow \mathrm{Diff}_{k-1}(E, F) \xrightarrow{i} \mathrm{Diff}_k(E, F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E, F)$$

is exact ([6], [11]).

4° Let $L \in \mathrm{Diff}_k(E, F)$. Then L^* exists and $L^* \in \mathrm{Diff}_k(F, E)$ (Proposition 2.4 and [12]).

Lemma 2.6. With the above notations we have the following:

(i) for each $L \in \mathrm{Diff}_k(E, F)$

$$\sigma_k(L)^* = \sigma_k(L^*),$$

where $\sigma_k(L)^*$: $\pi^*F \longrightarrow \pi^*E$ is the adjoint of $\sigma_k(L)$; $\pi^*E \longrightarrow \pi^*F$.

(ii) for $L_1 \in \mathrm{Diff}_k(E, F)$ and $L_2 \in \mathrm{Diff}_l(F, G)$

$$\sigma_{k+l}(L_2 \circ L_1) = \sigma_l(L_2) \circ \sigma_k(L_1),$$

where G is also a vector bundle over M.

Proof. We shall prove that $P \in \mathrm{Diff}_k(E, F)$ is a Pseudodifferential operator ([3], [4], [6]) from E to F. By Definition 2.2 for any choice of local coordinates, local trivializations and $f = (f_1, \cdots, f_p) \in [\mathscr{E}(U)]^p \cong \mathscr{E}(U, E|_U)$

$$P(f)_i = \sum_{|\alpha| \leq k} \sum_{j=1}^{p} a_\alpha^{ij} D^\alpha f_j \quad (i = 1, \cdots, q = \dim_C F)$$

Take a sufficiently small open subset $U' \subset\subset U$ such that there exists an open subset V of U satisfying the condition

$$U' \subset V \subset \bar{V} \subset U.$$

We define a C^∞-function

$$a_\alpha^{ij}: U \longrightarrow \boldsymbol{C}$$

such that

$$a_\alpha^{ij}(x)=\begin{cases} a_\alpha^{ij} & ,x\in U' \\ 0 & ,x\in U-V \\ 0\leq |a_\alpha^{ij}(x)|\leq |a_\alpha^{ij}|, & x\in U. \end{cases}$$

Then for each $x\in U'$

$$P(f)_i(x)=\sum_{|\alpha|\leq k}\sum_{j=1}^{p} a_\alpha^{ij}(x)\ D^\alpha\ f_j(x).$$

Define

$$p_\alpha^{ij}(x,\xi)=a_\alpha^{ij}(x)\ \xi^\alpha$$

for $x\in U$ and $\xi(\neq 0)\in \boldsymbol{R}^n$. Then $p_\alpha^{ij}(x,\xi)$ $(i=1,\cdots,q,\ j=1,\cdots,p,\ |\alpha|\leq k)$ is a C^∞-function satisfying the following:

(a) for each compact subset $K\subset U$ and multiindices β,γ there exists a constant $C_{\beta,\gamma,K}$ such that

$$|D_x^\gamma\ D_\xi^\beta\ p_\alpha^{ij}(x,\xi)|\leq C_{\beta,\gamma,K}(1+|\xi|^2)^{|\alpha|-|\beta|}\ (x\in K,\xi(\neq 0)\in \boldsymbol{R}^n),$$

where if $\xi=(\xi_1,\cdots,\xi_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$, then

$$D_\xi^\beta=(-i)^{|\beta|}\frac{\partial^{\beta_1}}{\partial \xi_1^{\beta_1}}\cdots\cdots\frac{\partial^{\beta_n}}{\partial \xi_n^{\beta_n}},$$

(b) for a fixed ξ $p_\alpha^{ij}(x,\xi)$ has a compact support in U about x,

(c) $\sigma_{|\alpha|}(p_\alpha^{ij})(x,\xi)=\lim_{\lambda\to\infty}\dfrac{p_\alpha^{ij}(x,\lambda\xi)}{\lambda^{|\alpha|}}=a_\alpha^{ij}(x)\ \xi^\alpha\ (\lambda>0).$

Therefore P is a Pseudo-differential operator from E to F.
By the well known theorem with respect to Pseudo-differential operators ([6]), [11]) we have

$$\sigma_k(P^*)=\sigma_k(P)^*$$

and

$$\sigma_{k+l}(L_2\circ L_1)=\sigma_l(L_2)\circ\sigma_k(L_1),$$

where $P,\ L_1\in \mathrm{Diff}_k(E,F)$ and $L_2\in \mathrm{Diff}_l(F,G)$. ///

3. Elliptic Operators

Let E and F be vector bundles over M.

Definition 3.1. A differential operator $L \in \mathrm{Diff}_k(E,F)$ is said to be *elliptic* if

$$\sigma_k(L)(x,v):\ E_x \longrightarrow F_x$$

is an isomorphism for all $(x,v) \in T'(M)$, where E_x is the fibre of E at $x \in M$.

Proposition 3.2. If $L \in \mathrm{Diff}_k(E,F)$ is elliptic, then so is $L^* \in \mathrm{Diff}_k(F,E)$, where L^* is the adjoint of L.

Proof. It suffices to prove that

$$\sigma_k(L)^*:\ \pi^*F \longrightarrow \pi^*E$$

is an isomorphism since $\sigma_k(L)^* = \sigma_k(L^*)$ by (a) of Lemma 2.6 (π: $T'(M) \longrightarrow M$ is a projection). Since for each $(x,v) \in T'(M)$

$$\sigma_k(L)\ (x,v):\ E_x \longrightarrow F_x$$

is an isomorphism it follows that

$$\sigma_k(L)^*(x,v):\ F_x \longrightarrow E_x$$

is also an isomorphism. ///

We have the following results ([4], [8], [11]):

1°. For each elliptic differential operator $L \in \mathrm{Diff}_k(E,F)$ there exists a differential operator $\tilde{L} \in \mathrm{Diff}_k(F,E)$ such that

$$L \circ \tilde{L} - 1_{\mathcal{E}(M,F)} \in OP_{-1}(F,F),$$
$$\tilde{L} \circ L - 1_{\mathcal{E}(M,E)} \in OP_{-1}(E,E)$$

where $\tilde{L}$ is called a parametrix of L.

2° $L \circ \tilde{L} - 1_{\mathcal{E}(M,F)}$ and $\tilde{L} \circ L - 1_{\mathcal{E}(M,E)}$ defined as in 1° are compact operator of order 0(see the commutative diagram

$$\begin{array}{ccc} W^s(M,E) & \xrightarrow{(\tilde{L} \circ L - 1_{\mathcal{E}(M,E)})_s} & W^s(M,E) \\ & {\scriptstyle (\tilde{L} \circ L - 1_{\mathcal{E}(M,E)})_s} \searrow \qquad \nearrow {\scriptstyle j} & \\ & W^{s+1}(M,E) & \end{array}$$

where the inclusion j is a compact operator).

3°. Let B be a Banach space (over $\boldsymbol{R}$ or $\boldsymbol{C}$) and let S: $B \longrightarrow B$ be a compact operator. Then for $T = 1_B - S$ the following hold:

(a) Ker $T = T^{-1}(0)$ is a finite dimensional vector space.

(b) $T(B)$ is closed in B and Coker $T=B/T(B)$ is a finite dimensional vector space.

By using 1°~3° above we can prove the next property.

Proposition 3.3. Let $L\in \mathrm{Diff}_k(E,F)$ be elliptic. Then Ker $L\subset \mathscr{E}(M,E)$ is a finite dimensional C-vector space.

Proof. There exists a parametrix $\tilde{L}$ of L such that $\tilde{L}\circ L-1_{\mathscr{E}(M,E)}\in OP_{-1}(E,E)$ which is a compact operator by 1° and 2°.

Since

$$1_{\mathscr{E}(M,E)}-(1_{\mathscr{E}(M,E)}-\tilde{L}\circ L)=\tilde{L}\circ L,$$

by 3°, Ker $\tilde{L}\circ L$ is a finite dimensional subspace of $\mathscr{E}(M,E)$. Since

$$\mathrm{Ker}\ L\subseteq \mathrm{Ker}\ \tilde{L}\circ L,$$

it is obvious that Ker L is a finite dimensional. ///

Let $E_0,\cdots,E_N$ be a sequence of vector bundles over M. Assume that there is a sequence of differential operator $L_0,\ L_1,\cdots,L_{N-1}$ with fixed order k as in the diagram:

$$E:\ 0\longrightarrow \mathscr{E}(M,E_0)\xrightarrow{L_0}\cdots\xrightarrow{L_{N-1}}\mathscr{E}(M,E_N)\xrightarrow{L_N}0$$

where $L_{-1}=0=L_N$. Thus we have the associated symbol sequence of E:

$$0\longrightarrow \pi^*(E_0)\xrightarrow{\sigma_k(L_0)}\cdots\xrightarrow{\sigma_k(L_{N-1})}\pi^*(E_N)\longrightarrow 0, \quad \cdots\cdots\cdots\cdots(3-1)$$

where π: $T'(M)\longrightarrow M$ is the projection.

Definition 3.4. In E, if $L_j\circ L_{j-1}=0$ for all $j=0,\cdots,N$, then E is called a complex of bundles over M. If (3—1) is an exact sequence, then E is called an *elliptic complex* of *bundles* over M.

Under the above notations, we assume that E is a complex of bundles over M. Then the cohomology $H^q(E)$ $(q=0,\ 1,\cdots,N)$ of E are defined by

$$H^q(E)=\frac{\mathrm{Ker}(L_q:\ \mathscr{E}(M,E_q)\to\mathscr{E}(M,E_{q+1}))}{\mathrm{Im}(L_{q-1}:\ \mathscr{E}(M,E_{q-1})\to\mathscr{E}(M,E_q))}=\frac{Z^q(E)}{B^q(E)}. \quad\cdots\cdots(3-2)$$

Moreover, if E is elliptic, then the Laplacian operators $\triangle_j$ $(j=0,\ 1,\cdots,N)$ are defined by

$$\triangle_j=L_j^*L_j+L_{j-1}\ L_{j-1}^*:\ \mathscr{E}(M,E_j)\longrightarrow\mathscr{E}(M,E_j), \quad\cdots\cdots\cdots\cdots(3-3)$$

where L_j^*: $\mathscr{E}(M,E_{j+1})\longrightarrow\mathscr{E}(M,E_j)$ is adjoint of L_j: $\mathscr{E}(M,E_j)\longrightarrow\mathscr{E}(M,E_{j+1})$.

Proposition 3.5. With the above notations the Laplacian operators $\triangle_j(j=0,\cdots,N)$

are elliptic differential operators of order $2k$.

Furthermore, $\triangle_j = \triangle_j^*$(i.e., $\triangle_j$ is a *self-adjoint operator*).

Proof. That $\triangle_j$ is a differential operator of order $2k$ is clear by 4° in §2. Moreover, by Lemma 2.6

$$\sigma_{2k}(\triangle_j) = \sigma_k(L_j)^* \circ \sigma_k(L_j) + \sigma_k(L_{j-1}) \circ \sigma_k(L_{j-1})^*$$

Suppose the diagram

$$\pi^* E_{j-1} \underset{\sigma_k(L_{j-1})^*}{\overset{\sigma_k(L_{j-1})}{\rightleftarrows}} \pi^* E_j \underset{\sigma_k(L_j)^*}{\overset{\sigma_k(L_j)}{\rightleftarrows}} \pi^* E_{j+1}.$$

Since (3—1) is exact $\mathrm{Im}(\sigma_k(L_{j-1})) = \mathrm{Ker}(\sigma_k(L_j))$.

On the other hand, it follows from

$$\mathrm{Ker}(\sigma_k(L_j)) \perp \mathrm{Im}(\sigma_k(L_j)^*)$$

that

$$\pi^* E_j = \mathrm{Ker}(\sigma_k(L_j)) \oplus \mathrm{Im}(\sigma_k(L_j)^*).$$

Furthermore

$$\sigma_k(L_{j-1}) \circ \sigma_k(L_{j-1})^* \mid \mathrm{Ker}\ \sigma_k(L_j) \text{ and}$$
$$\sigma_k(L_j)^* \circ \sigma_k(L_j) \mid \mathrm{Im}(\sigma_k(L_j))^*$$

are injective homomorphisms, and it follows that

$$\sigma_{2k}(\triangle_j) = I_{\pi^* E_j}.$$

Accordingly, $\triangle_j$ is an elliptic operator.

Next, for all $u, v \in \mathscr{E}(M, E_j)$

$$\begin{aligned}(\triangle_j u, v) &= ((L_j^* \circ L_j + L_{j-1} \circ L_{j-1}^*)u, v) \\ &= (L_j^* \circ L_j\, u, v) + (L_{j-1} \circ L_{j-1}^*\, u, v) \\ &= (u, (L_j^* \circ L_j + L_{j-1} \circ L_{j-1}^*)\, v) \\ &= (u, \triangle_j v)\end{aligned}$$

and thus $\triangle_j = \triangle_j^*$, where (,)means that for $\xi, \eta \in \mathscr{E}(M, E_j)$

$$(\xi, \eta) = \int_M < \xi(x),\ \eta(x) >_{E_j} dx$$

(see (2—1) in §2) and $< , >_{E_j}$ is the given Hermitian metrics on E_j. ///

Let $L\in\mathrm{Diff}_k(E,E)$ be a self-adjoint elliptic operator. Then there are two linear mappings

$$H_L:\ \mathscr{E}(M,E)\longrightarrow \mathrm{Ker}\ L,\quad G_L:\ \mathscr{E}(M,E)\longrightarrow(\mathrm{Ker}\ L)^{\perp}$$

satisfying the following ([7], [10], [13]):

4°. $L\circ G_L+H_L=G_L\circ L+H_L=1_{\mathscr{E}(M,E)}$

5°. $H_L,\ G_L\in OP_{-1}$, which extend to a bounded operator on $W^0(E,E)$

6°. $\mathscr{E}(M,E)=\mathrm{Ker}\ L\oplus G_L\circ L(\mathscr{E}(M,E))$
$$=\mathrm{Ker}\ L\oplus L\circ G_L(\mathscr{E}(M,E))$$

and this decomposition is orthogonal with respect to the inner product (,) in $W^0(E,E)$. (Note that E is a vector bundle over M). In the above situation, the operator G_L is called the *Green's operator* of L and for simplicity we shall put

$$\mathrm{Ker}\ L=\mathscr{H}_L.$$

As before, let

$$E:\ 0\longrightarrow\mathscr{E}(M,E_0)\xrightarrow{L_0}\mathscr{E}(M,E_1)\longrightarrow\cdots\xrightarrow{L_{N-1}}\mathscr{E}(M,E_N)\xrightarrow{L_N}0$$

be an elliptic complex of bundles over M, where

$$L_j\in\mathrm{Diff}_k(E_j,\ E_{j+1})\ \ (j=0,\cdots,N-1).$$

Lemma 3.6. Under the above notations the following hold.

(i) For $\xi\in\mathscr{E}(M,E_j)$

$$\triangle_j\xi=0\Longleftrightarrow L_j\,\xi=0=L_{j-1}^*\,\xi\ \ (j=0,\cdots,N).$$

(ii) $L_jG_{\triangle_j}=G_{\triangle_{j+1}}\,L_j$

Proof. (i): $(\triangle_j\xi,\xi)=(L_j^*L_j\xi,\xi)+(L_{j-1}^*\,\xi,\ L_{j-1}^*\,\xi)$
$$=\|L_j\xi\|^2+\|L_{j-1}^*\,\xi\|^2,$$
where (,) is the inner product on $\mathscr{E}(M,E_i)$ $(i=j-1,j,j+1)$. Therefore

$$\triangle_j\xi=0\Longleftrightarrow\|L_j\xi\|=0=\|L_{j-1}^*\,\xi\|$$
$$\Longleftrightarrow L_j\,\xi=0=L_{j-1}^*\,\xi$$

(ii): For simplicity we put

$$G_{\triangle_j}=G_j,\ \ H_{\triangle_j}=H_j\ \text{and}\ \mathscr{H}_{\triangle_j}=\mathscr{H}_j$$

Since it is obvious that

$$G_j|_{\mathscr{H}_j}=L_j|_{\mathscr{H}_j}=0$$

by (i) and 6°, we shall prove that

$$L_j G_j=G_{j+1}L_j \text{ on } \mathscr{H}_j^{\perp}.$$

Take $\varphi\in\mathscr{E}(M,E_j)$ and put $\triangle_j\ \varphi=\xi$.
From 4° we get

$$\begin{aligned} L_j\ \varphi&=H_{j+1}(L_j\ \varphi)+G_{j+1}\ \triangle_{j+1}(L_j\ \varphi)\\ &=H_{j+1}\ L_j\ \varphi+G_{j+1}\ L_j\ \triangle_j\ \varphi \end{aligned}$$

by definition of $\triangle_j$ (i.e., $\triangle_{j+1}\ L_j=L_j\ \triangle_j$).
On the other hand, since

$$\varphi=H_j\ \varphi+G_j\ \triangle_j\ \varphi \text{ (by 4°)},$$

we have

$$L_j\varphi=L_j\ H_j\ \varphi+L_j\ G_j\ \triangle_j\ \varphi.$$

It follows that

$$H_{j+1}\ L_j\ \varphi+G_{j+1}\ L_j\ \triangle_j\ \varphi=L_j\ H_j\ \varphi+L_j\ G_j\ \triangle_j\ \varphi.$$

Thus we have

$$G_{j+1}\ L_j\ \xi=L_j\ G_j\ \xi,$$

because $H_{j+1}\ L_j=L_j\ H_j=0$. ///

Theorem 3.7. Under the above notations, for each $q(q=0,1,\cdots,N)$ there is a cannonical isomorphism

$$H^q(E)\cong\mathscr{H}_q\ (=\text{Ker}\ \triangle_q).$$

Proof. From (3—2) $H^q(E)=Z^q(E)/B^q(E)$, where $Z^q(E)=\text{Ker}\ L_q$ and $B^q(E)=\text{Im}\ L_{q-1}$.
Define

$$\begin{array}{ccc} \Phi:\ Z^q(E) & \longrightarrow & \mathscr{H}_q \\ \Cup & & \Cup \\ \xi & \longmapsto & \Phi(\xi)=H_q(\xi) \end{array}$$

Then, it follows from (i) of Lemma 3.6 that Φ is linear and surjective.
We shall prove that

$$\text{Ker}\ \Phi=B^q(E)$$

Assume that $\xi \in Z^q(E)$ and $H_q(\xi)=0$, then we have

$$\begin{aligned}\xi &= H_q(\xi)+\triangle_q\, G_q(\xi) \quad (\text{by } 4^\circ)\\ &= H_q(\xi)+(L_q^*\, L_q+L_{q-1}\, L_{q-1}^*)\, G_q(\xi) \quad (H_q(\xi)=0)\\ &= L_q^*\, G_{q+1}\, L_q(\xi)+L_{q-1}\, L_{q-1}^*\, G_q(\xi) \quad ((\text{ii}) \text{ of Lemma } 4.6)\\ &= L_{q-1}\, L_{q-1}^*\, G_q(\xi) \quad (L_q(\xi)=0)\\ &= L_{q-1}(L_{q-1}^*\, G_q(\xi)) \in B^q(E). \quad ///\end{aligned}$$

4. The Laplacian $\triangle$ and Its Eigenvalues

Throughout this section, by X we mean a compact oriented Riemannian manifold with dimension n. As is well known there is the *de Rham complex*

$$E(X):\ 0 \longrightarrow E^0(X) \xrightarrow{d} E^1(X) \longrightarrow \cdots \xrightarrow{d} E^n(X) \longrightarrow 0,$$

where

$$E^p(X)=\{s:\ X \longrightarrow \Lambda^p\, T^*(X) \mid s \text{ is a } C^\infty\text{-section}\},$$

$\Lambda^p\, T^*(X)$ is the set of all p-forms on X and d is the usual exterior differentiation operator ([5]).

Since $d \circ d=0$ and each $\Lambda^p T^*(X)$ is a $\boldsymbol{R}$-vector bundle over X with dimension ${}_nC_p$, $E(X)$ is a complex of bundles over X (See (3—1) of §3) (Note that d is a differential operator of order 1).

Proposition 4.1. $E(X)$ is an elliptic complex.

Proof. For $T'(X)=T^*(X)-X$, let $\pi:\ T'(X) \longrightarrow X$ be the projection. we have to prove that

$$0 \longrightarrow \pi^*\, \Lambda^0 T^*(X) \xrightarrow{\sigma_1(d)} \cdots \xrightarrow{\sigma_1(d)} \pi^*\, \Lambda^n\, T^*(X) \longrightarrow 0$$

is exact. It suffices to prove that

$$\pi^*\, \Lambda^{p-1}\, T^*(X) \xrightarrow{\sigma_1(d)} \pi^*\, \Lambda^p\, T^*(X) \xrightarrow{\sigma_1(d)} \pi^*\, \Lambda^{p+1} T^*(X) \quad \cdots\cdots\cdots (4\text{—}1)$$

is exact at $\pi^*\, \Lambda^p T^*(X)$. For each $(x, v) \in T'(X)$ and $e \in \Lambda^p\, T^*(X)$

$$\begin{aligned}\sigma_1(d)\ (x, v)\ e &= d(g-g(x))(x)\ \Lambda\ e\\ &= \left(\frac{\partial g}{\partial x_1}(x)dx_1+\cdots+\frac{\partial g}{\partial x_n}(x)\, dx_n\right)\Lambda\ e\\ &= v\Lambda e (\in \Lambda^{p+1}\, T^*(X))\end{aligned}$$

where $dg_* = v$ (see Definition 2.5). Suppose

$$\sigma_1(d)\ (x,v)\ e = v\Lambda e = 0\ (e \in \Lambda^p T^*(X))_x).$$

Then there exists an element $e' \in (\Lambda^{p-1} T^*(X))_x$ such that $v\Lambda e' = e$. It follows from $\sigma_1(d)$ $(x,v)e' = v\Lambda e'$ that $e \in \sigma_1(d)(x,v)(\Lambda^{p-1} T^*(X))_x$, and thus (4—1) is exact. ///

We shall define the Laplacian $\triangle_j (j=0,1,\cdots,n)$ on the elliptic complex $E(X)$ as follows. The linear operator

$$*: E^p(X) \longrightarrow E^{n-p}(X)\ \ (p=0,1,\cdots,n)$$

is defined so that

$$** = (-1)^{p(n-p)}\ 1_{E^p(X)}.$$

We also define the linear map

$$\delta: E^p(X) \longrightarrow E^{p-1}(X)\ \ (p=0,\cdots,n+1)$$

by $\delta = (-1)^{n(p+1)+1} *d*$, where $E^{-1}(X) = 0 = E^{n+1}(X)$.

Definition 4.2. We define the inner product $< , >_p$ on $E^p(X)$ by setting

$$<\alpha,\beta>_p = \int_X \alpha\Lambda*\beta,\ \ \alpha,\beta \in E^p(X)\ \ (p=0,1,\cdots,n)$$

(Note that our integral is well defined since X is oriented).

Proposition 4.3. Under the inner product $< , >_p$ the operator δ is the adjoint of d.

Proof. For $\alpha \in E^p(X)$ and $\beta \in E^{p+1}(X)$ we shall prove that

$$<d\alpha,\ \beta>_{p+1} = <\alpha,\ \delta\beta>_p.$$

Since

$$\begin{aligned} d(\alpha\Lambda*\beta) &= d\alpha\ \Lambda*\beta + (-1)^p \alpha\ \Lambda d*\beta \\ &= d\alpha\ \Lambda*\beta - \alpha\ \Lambda*\delta\beta, \end{aligned}$$

we have

$$\int_X d(\alpha\,\Lambda*\beta) = \int_X d\alpha\,\Lambda*\beta - \int_X \alpha\,\Lambda*\delta\beta.$$

Since $\alpha\,\Lambda*\beta$ is a smooth $(n-1)$-form on a compact differentiable manifold X with $\dim_R X = n$,

$$\int_X d(\alpha \Lambda * \beta) = 0$$

by a Stoke's theorem ([1],[7],[13]).
Therefore we have

$$<d\alpha,\ \beta>_{p+1} = \int_X d\alpha \Lambda * \beta = \int_X \alpha \Lambda * \delta\beta = <\alpha,\ \delta\beta>_p.$$

Crollary 4.4. For each $p (p=0, 1, \cdots, n)$

$$\triangle_p = \delta d + d\delta :\ E^p(X) \longrightarrow E^p(X)$$

is the Laplacian operator, and thus

$$H^p(E) \cong \text{Ker}\ \triangle_p,$$

where $H^p(E)$ is just the p-th de Rham cohomology group of X.

Proof. Since $E(X)$ is elliptic (Proposition 4.1) and δ is the adjoint of d, it is clear that $\triangle_p$ is the Laplacian operator (see 3—3).
Moreover, since $\triangle_p$ is also self-adjoint and elliptic, it follows from Theorem 3.7 that

$$H^p(E) \cong \text{Ker}\ \triangle_p. \quad ///$$

Each element of Ker $\triangle_p$ is called a $\triangle_p$-harmonic section.
Then, Corollary 4.4 says that there is an one-to-one correspondence between the set of all $\triangle_p$-harmonic sections and the set of all p-th de Rham cohomology classes, which is the classical Hodge theorem.
In view of this fact, Theorem 3.7 is a generalization of the classical Hodge theorem.

In the sequel, we shall put

$$\triangle_p = \triangle,\ G_{\triangle_p} = G,\ H_{\triangle_p} = H,\ \text{Ker}\triangle_p = \mathscr{H}_\triangle$$

for simplicity.

Definition 4.5. (i) A linear bounded map

$$l :\ E^p(X) \longrightarrow \boldsymbol{R}$$

is called a weak solution of $\triangle\omega = \alpha$ ($\alpha \in E^p(X)$) if for every $\varphi \in E^p(X)$

$$l(\triangle^* \varphi) = <\alpha,\ \varphi>_p.$$

(Note that for each weak solution $l :\ E^p(X) \longrightarrow \boldsymbol{R}$ of $\triangle\omega = \alpha$ there is always an element

$\omega \in E^p(X)$ such that $l(\beta)=<\omega, \beta>_p$ for all $\beta \in E^p(X)$ ([11]) (4—2).

(ii) A real number λ such that there exists an identically nonzero element $u \in E^p(X)$ which $\triangle u=\lambda u$ is called an eigenvalue of $\triangle$. If λ is an eigenvalue of $\triangle$ and $\triangle u=\lambda u$, then u is called an eigenfunction of λ.

For each eigenvalue λ of $\triangle$,

$$\{u \in E^p(X) \mid \triangle u=\lambda u\}$$

is called the eigenspace of λ.

Proposition 4.6. (i) Each eigenvalue of $\triangle$ is nonnegative.

(ii) Each eigenspace of $\triangle$ is finite dimensional.

(iii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Proof. At first, we have to note that

(a) G commutes with d, δ, and $\triangle$

(b) $^\forall\alpha \in E^p(X)$ $\alpha=H\alpha+\delta d\, G\alpha+d\delta\, G\alpha$

(see 4°~5° and Lemma 3.6 in §3).

(i) Assume that $\triangle u=\lambda u$, $\lambda \neq 0$, and $u \in \mathscr{H}_\triangle^\perp$, then

$$\begin{aligned} <u,u> &= <G\triangle u, u> \\ &= \lambda <Gu, u> \\ &= \lambda <Gu, (\delta dG+d\delta G)\, u> \\ &= \lambda\{<dGu, dGu>+<\delta Gu, \delta Gu>\} \\ &= \lambda(\|dGu\|^2+\|\delta Gu\|^2) \\ &> 0, \end{aligned}$$

and thus $\lambda>0$, where $< , > = < , >_p$.

If $\mathscr{H}_\triangle \neq \{0\}$, then for $u(\neq 0) \in \mathscr{H}_\triangle$ $\triangle u=0u$. That is, 0 is an eigenvalue of $\triangle$. It follows that each eigenvalue of $\triangle$ is nonnegative.

(ii) Let $\lambda(\neq 0)$ be an eigenvalue of $\triangle$.

Then

$$\triangle-\lambda:\ E^p(X) \longrightarrow E^p(X)$$

is a differential operstor of order 2. Since

$$\begin{aligned} \sigma_2(\triangle-\lambda)(x,v)(e) &= (\triangle-\lambda)\left(\frac{(-1)^2}{2!}(g(x)-g)^2 f\right)(x) \\ &= \triangle\left(\frac{(-1)^2}{2!}(g(x)-g)^2 f\right)(x) \\ &= \sigma_2(\triangle)\ (x,v)\ e \end{aligned}$$

and $\sigma_2(\triangle)(x,v)$ is an isomorphism, so is $\sigma_2(\triangle-\lambda)(x,v)$, where $(x,v)\in T'(X)$, $dg_x=v$ and $f(x)=e$.

It follows that $\triangle-\lambda$ is a self-adjoint elliptic differential operator.

It follows from Proposition 3.3 that Ker $(\triangle-\lambda)$ is finite dimensional.

Therefore,

$$\mathrm{Ker}(\triangle-\lambda)=\{u\in E^p(X)\mid \triangle u=\lambda u\}$$

is finite dimensional. When $\lambda=0$,

$$\mathscr{H}_\triangle=\mathrm{Ker}\ \triangle=\{u\in E^p(X)\mid \triangle u=0\}$$

is finite dimensional.

(iii) Let $\lambda_1\neq\lambda_2$ (both nonzero) be two eigenvalues of $\triangle$.

Assume that $\triangle u_1=\lambda_1 u_1$ and $\triangle u_2=\lambda_2 u_2(u_1,\ u_2\in E^p(X))$, then

$$\begin{array}{ccc}<\triangle u_1,\ u_2> & = & <u_1,\ \triangle u_2> \\ \| & & \| \\ \lambda_1<u_1,\ u_2> & = & \lambda_2<u_1,\ u_2>\end{array}$$

and thus $<u_1,\ u_2>=0$, where $<\ ,\ >=<\ ,\ >_p$.

When $\lambda_1=0$,

$$0=<\triangle u_1,\ u_2>=\lambda_2<u_1,\ u_2>\ (\lambda_2\neq 0,)$$

and thus $<u_1,\ u_2>=0$, where $<\ ,\ >=<\ ,>_p$. ///

Theorem 4.7. There are eigenvalues $0\leq\lambda_1\leq\lambda_2\cdots$ of $\triangle$. If an orthonormalized sequence of eigenfunctions $\{u_i\}$ corresponds to $0\leq\lambda_1\leq\lambda_2\leq\cdots$ where each eigenvalue is included as many times as the dimension of its eigenspace, then for all $\alpha\in E^p(X)$

$$\lim_{n\to\infty}\|\alpha-\sum_{i=1}^{n}<\alpha,u_i>_p u_i\|=0,$$

where $\|\alpha\|=<\alpha,\alpha>_p^{\frac{1}{2}}$

Proof. For simplicity we also put $<\ ,\ >_p=<\ ,\ >$.

Consider two linear maps:

$$\triangle:\ \mathscr{H}_\triangle^{\perp}\longrightarrow\mathscr{H}_\triangle^{\perp},\ G:\ \mathscr{H}_\triangle^{\perp}\longrightarrow\mathscr{H}_\triangle^{\perp}$$

(G is the Green's operator of $\triangle\mid\mathscr{H}_\triangle^{\perp}$). Since for each $\alpha\in\mathscr{H}_\triangle^{\perp}$

$$\triangle G\alpha=G\triangle\alpha=\alpha\ \text{(see } 4^\circ \text{ in §3)},$$

if $\triangle\alpha=\lambda\alpha(\lambda>0)$ then $G\alpha=\frac{1}{\lambda}\alpha$, *i.e.*, if $\lambda(\neq 0)$ is an eigenvalue of $\triangle$ then $\frac{1}{\lambda}$ is an eigenvalue of G.

We put

$$\eta_1=\sup_{\substack{\varphi\in\mathscr{H}_\triangle^+ \\ \|\varphi\|=1}} \|G\varphi\|.$$

Then it is obvious that $\|G\|=\eta_1$, where $\|G\|$ is the norm of G as an operator. Therefore we see that $\eta_1>0$ and for every $\varphi\in\mathscr{H}_\triangle^+$ $\|G\varphi\|\leq\eta_1\|\varphi\|$. (4—3)

Let $\{\varphi_j\}\subset\mathscr{H}_\triangle^+$ be a maximizing sequence for η_1, that is, $\|\varphi_j\|=1$ and $\|G\varphi_j\|\longrightarrow\eta_1$. Then

$$\begin{aligned}\|G^2\varphi_j-\eta_1^2\varphi_j\|^2&=<G^2\varphi_j-\eta_1^2\varphi_j,\ G^2\varphi_j-\eta_1^2\varphi_j>\\&=\|G^2\varphi_j\|^2-2\eta_1^2<G^2\varphi_j,\ \varphi_j>+\eta_1^4\\&\leq\eta_1^2\|G\varphi_j\|^2-2\eta_1^2\|G\varphi_j\|^2+\eta_1^4\end{aligned}$$

(Note that $<G^2\varphi_j,\ \varphi_j>=<G^2\varphi_j,\ \triangle G\varphi_j>=<G\varphi_j,\ G\varphi_j>$ for all $\varphi_j\in\mathscr{H}_\triangle^+$), and thus

$$\lim_{j\to\infty}\|G^2\varphi_j-\eta_1^2\varphi_j\|=0.$$

Therefore, if we put $\Psi_j=G\varphi_j-\eta_1\varphi_j$ then

$$\begin{aligned}\eta_1\|\Psi_j\|^2&\leq<\Psi_j,\ G\Psi_j>+\eta_1\|\Psi_j\|^2=<\Psi_j,\ G\Psi_j+\eta_1\Psi_j>\\&=<\Psi_j,\ G^2\varphi_j-\eta_1^2\varphi_j>\end{aligned}$$

(Note that $<\Psi_j,\ G\Psi_j>=<(d\delta G+\delta dG)\Psi_j,\ G\Psi_j>$
$=<\delta G\Psi_j,\ \delta G\Psi_j>+<dG\Psi_j,\ dG\Psi_j\geq 0)$,

and thus $\|\Psi_j\|=\|G\varphi_j-\eta_1\varphi_j\|\longrightarrow 0$. This implies that

$$\lim_{j\to\infty}G\varphi_j=\lim_{j\to\infty}\eta_1\varphi_j.$$

Since $\|G\varphi_j\|\longrightarrow\eta_1$, there is a subsequence of $\{\varphi_j\}$, call it $\{\varphi_j\}$, such that $\{G\varphi_j\}$ is a Cauchy sequence (Note that $E^p(X)$ is a Hilbert space under the inner product $<\ ,\ >_p$).

Define a linear map

$$l:\ E^p(X)\longrightarrow\boldsymbol{R}$$

by $l(\beta)=\lim_{j\to\infty}\eta_1<G\varphi_j,\ \varphi>$, where $\beta\in E^p(X)$.

Then it is clear that l is bounded. For $\alpha\in E^p(X)$, we have

$$l((\triangle-\frac{1}{\eta_1})^*\alpha)=\lim_{j\to\infty}\eta_1<G\varphi_j,(\triangle-\frac{1}{\eta_1})^*\alpha>$$
$$=\lim_{j\to\infty}\{<\eta_1\triangle G\varphi_j,\alpha>-\eta_1<G\varphi_j,\frac{\alpha}{\eta_1}>\}$$
$$=\lim_{j\to\infty}\{<\eta_1\varphi_j,\alpha>-\eta_1<\eta_1\varphi_j,\frac{\alpha}{\eta_1}>\}$$
$$=0$$
$$=<0,\alpha>.$$

Since $\triangle-1/\eta_1$ is an elliptic operator, by (i) of Definition 4.5, ℓ is a weak solution of $(\triangle-1/\eta_1)u=0$. Therefore, by (4—2) there exists an element $u\in E^p(X)$ such that $(\triangle-1/\eta_1)\,u=0$. Hence, $\triangle u=(1/\eta_1)\,u$ and $1/\eta_1=\lambda_1$ is an eigenvalue of $\triangle$.

Let u_1 be the eigenfunction of the eigenvalue $\lambda_1=1/\eta_1$ $(u_1\in\mathscr{H}_\triangle^\perp)$ and let R_1 be the $\boldsymbol{R}$-vector space generated by u_1. We put

$$\eta_2=\sup_{\substack{\varphi\in(\mathscr{H}_\triangle\oplus R_1)^\perp\\ ||\varphi||=1}}||G\varphi||,$$

then by the above way we can prove that $0<\eta_2\leq\eta_1$ and $\lambda_2=1/\eta_2$ is an eigenvalue of $\triangle$ (Note that $\lambda_1\leq\lambda_2$). Repeating this way we can get eigenvalues $0<\lambda_1\leq\lambda_2\leq\cdots$. We rearrange this sequence to get a sequence $0\leq\lambda_1\leq\lambda_2\leq\cdots$ of eigenvalues where each eigenvalue is included as many times as the dimension of its eigenspace.

Let $\{u_i\}$ be an orthonormalized sequence of eigenfunctions corresponding $0\leq\lambda_1\leq\lambda_1\cdots$. Assume that

$$\dim_R\,(\mathscr{H}_\triangle)=k.$$

Then, for each $\alpha\in E^p(X)$

$$H(\alpha)=\sum_{i=1}^{k}<\alpha,u_i>\,u_i,$$

and

$$\alpha-\sum_{i=1}^{k}<\alpha,u_i>u_i=\alpha-H(\alpha)\in\mathscr{H}_\triangle^\perp$$

Since $G|_{\mathscr{H}_\triangle^\perp}:\mathscr{H}_\triangle^\perp\longrightarrow\mathscr{H}_\triangle^\perp$ is surjective, there exists $\beta\in\mathscr{H}_\triangle^\perp$ such that

$$G\beta=\alpha-\sum_{i=1}^{k}<\alpha,u_i>\,u_i.$$

It follows that for $n<k$

$$\|\alpha-\sum_{i=1}^{n}<\alpha,\ u_i>u_i\|=\|G(\beta-\sum_{i=k+1}^{n}<\beta,\ u_i>u_i)\|$$

$$\leq\eta_{n+1}\|\beta-\sum_{i=k+1}^{n}<\beta,\ u_i>u_i\|$$

$$\leq\frac{1}{\lambda_{n+1}}\|\beta\|,$$

where ⓐ $<\beta-\sum_{i=k+1}^{n}<\beta,\ u_i>u_i,\ \beta-\sum_{i=k+1}^{n}<\beta,\ u_i>u_i>$

$$=\|\beta\|^2-\sum_{i=k+1}^{n}<\beta,\ u_i>^2,$$

ⓑ for the space R_n which is generated by $\{u_1,\cdots,u_n\}$

$$\eta_{n+1}=\sup_{\substack{\|\varphi\|=1\\ \varphi\in(\mathscr{X}_\Delta\oplus R_n)^\perp}}\|G\varphi\|$$

and $\|G\varphi\|\leq\eta_{n+1}\|\varphi\|$ by (4—3).

Since $\lambda_{n+1}\longrightarrow\infty$ as $n\longrightarrow\infty$ we have

$$\lim_{n\to\infty}\frac{1}{\lambda_{n+1}}\|\beta\|=0$$

and thus

$$\lim_{n\to\infty}\|\alpha-\sum_{i=1}^{\infty}<\alpha,\ u_i>u_i\|=0.\quad ///$$

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