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C*-Algebras and *-Representations

by

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0. Introduction

Theory of C^*-algebras is a branch of classical analysis and it has been developed since approximately 1900. C^*-algebras could be characterized as a special class of Banach algebras by means of a simple system of axioms and play a basic role in the study of the representations of a very extensive class of involutive Banach algebras. The study of C^*-algebras consists of two parts: one is related to the inherent structure of algebras ([1], [3], [12], [13]) and the other is concerned with the representations of a C^*-algebra ([5], [8], [12], [17]).

The purpose of this paper is to prove some properties of C^*-algebras without unity (section 3) and to study on *-representation (section 4).

In section 1, we describe some important terminologies and notations which are needed in section 3 and 4. In section 2, we construct three kinds of C^*-algebras in Example 2.1. It is noted that the third example is in connection with the Theorem 3.4. The purpose of section 3 is to prove Theorem 3.4 which states a property of C^*-algebras without unity.

Proposition 3.1, Lemma 3.2 and Lemma 3.3 are preliminaries to the proof of Theorem 3.4. Theorem 3.4 states that if M is a modular maximal ideal space of a C^*-algebra A and $C^\circ(\mathcal{M})$ is a set of bounded continuous functions on $\mathcal{M}$ vanishing at infinity, then a map $A \longrightarrow C^\circ(\mathcal{M})$ $(x \longrightarrow \hat{x})$ is isometric, surjective and *-isomorphic.

In section 4, we prove two results (Theorem 4.6 and Theorem 4.8) on *-representation and it is particularly observed that

$$\Sigma_A(a^*a) = \overline{W(\varphi(a^*a))}.$$

1. Preliminaries

Let A be a Banach algebra. A mapping $x \longrightarrow x^*$ of A into A is called an ***involution on** A* if it satisfies the following properties, for all $x, y \in A$ and $\lambda \in \boldsymbol{C}$:

$$(x+y)^*=x^*+y^*$$
$$(\lambda x)^*=\bar{\lambda}x^*$$
$$(xy)^*=y^*x^*$$
$$x^{**}=x.$$

where $\boldsymbol{C}$ is the set of all complex numbers.

A Banach algebra with an involution is sometimes called a *Banach *-algebra.* In a Banach * algebra A, if for all $x\in A$

$$\|x^*x\|=\|x\|^2$$

holds, then A is called a *C^*-algebra.*

Let A be a C^*-algebra. For each $x\in A$,

$\gamma_A(x)$ (or $\gamma(x)$) denotes the spectral radius of x, and
$\sigma_A(x)$ (or $\sigma(x)$) denotes the spectrum of x.

The followiog properties are well-known ([1], [3], [12], [13]).

1° For all $x\in A$, $\|x^*\|=\|x\|$

2° If A has a unity and $xx^*=x^*x$ (i.e, x is normal) ($x\in A$), then $\gamma_A(x)=\|x\|$.

3° If A has a unity and $x=x^*$ ($x\in A$) (i.e., x is hermitian or self-adjoint) then $\sigma_A(x)\subset\boldsymbol{R}$.

For *a* C^*-algebra A without unity, let A_1 consist of all ordered pairs (x,α), where $x\in A$ and $\alpha\in\boldsymbol{C}$. Define the vector space operations in A componentwise, define the multiplication in A_1 by

$$(x,\alpha)\ (y,\beta)=(xy+\alpha y+\beta x,\ \alpha\beta).$$

and define

$$\|(x,\alpha)\|=\|x\|+|\alpha|,\quad e_{A_1}=(0,1).$$

Then $(A_1, \|\ \|)$ is a Banach algebra with unity e_{A_1} ([12]).

In this case, we can regard A as a subalgebra of A_1 and as an ideal of A_1.

Therefore, for each $x\in A_1$ the mapping

$$T_X\colon A\longrightarrow A\ (\forall a\in A,\ T_X(a)=xa)$$

is a linear bounded operator. We put, for each $x\in A_1$,

$$|x| = \|T_x\|,$$

then the following is known ([1],[15],[16]).

4° For each $(a, \alpha) \in A_I$ we put

$$(a, \alpha)^* = (a^*, \bar{\alpha}):$$

(a) $(A_I, |\ |)$ is a C^*-algebra with unity.

(b) For all $a \in A$, $\|a\| = |a|$.

(c) Embedding $(A, \|\ \|) \longrightarrow (A_I, |\ |)$ is isometric.

$(A_I, |\ |)$ (or A_I) is called a *C*-algebra unitification* of A.

Furthermore A is a closed linear subspace of A_I.

Let A and B be C^*-algebras (both with unity or both without unity).

An algebra homomorphism

$$\varphi: A \longrightarrow B$$

is called a *-homomorphism if for all $a \in A$, $\varphi(a^*) = \varphi(a)^*$.

Proposition 1.1. If A and B are C^*-algebras with unity, and if $\varphi: A \longrightarrow B$ is a *-homomorphism such that $\varphi(e_A) = e_B$, then $\|\varphi(a)\| \leq \|a\|$ for all a in A and φ is continuous, where e_A and e_B are unities of A and B, respectively.

Proof. If $a - \lambda e_A$ is invertible in A, then $\varphi(a - \lambda e_A) = \varphi(a) - \lambda e_B$ is invertible in B. Hence $\sigma_B(\varphi(a)) \subset \sigma_A(a)$ and $\gamma_B(\varphi(a)) \leq \gamma_A(a)$.

If a is self-adjoint then so is $\varphi(a)$, and 2° yields

$$\|\varphi(a)\| = \gamma_B(\varphi(a)) \leq \gamma_A(a) = \|a\|.$$

If $a \in A$ is arbitrary then a^*a is self-adjoint, therefore

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| \leq \|a^*a\| = \|a\|^2,$$

thus $\|\varphi(a)\| \leq \|a\|$. ///

In particular, the following property is well-known ([1],[12])

5° Let A and B be C^*-algebras with unity, and let $\varphi: A \longrightarrow B$ be a *-homomorphism and injective. Then φ is isometric.

2. Examples

In this section we consider three illustrations of C^*-algebras.

Example 2.1. (i) Let T be a nonempty set. We put

$B(T)=\{x:\ T\longrightarrow \mathbf{C}\,|\,x$ is bounded, i.e., $\exists M s.t. |x(t)|\leq M\ \forall t\in T\}$

and define

$$\|x\|_\infty=\sup_{t\in T}|x(t)|$$

for all $x\in B(T)$. Then $B(T)$ is a commutative C^*-algebra with unity.

Proof. For all $x,y\in B(T)$ and $\lambda\in \mathbf{C}$ define the followings:

$$(x+y)(t)=x(t)+y(t),\quad xy(t)=x(t)y(t).$$
$$(\lambda x)(t)=\lambda x(t),\quad (t\in T).$$

Then, with the given norm $\|\ \|_\infty$ it is clear that $B(T)$ is a normed algebra over $\mathbf{C}$.

Let $\{x_n|n=1,2,\ldots\}$ be a Cauchy sequence in $B(T)$. i.e., for each $\varepsilon>0$ there exists $N>0$ such that for $n,m\geq N$

$$\|x_n-x_m\|_\infty<\varepsilon.$$

This means that for all $t\in T$ $|x_n(t)-x_m(t)|<\varepsilon$. Therefore, for each $t\in T$ $\{x_n(t)|n=1,2,\ldots\}$ is a Cauchy sequence in $\mathbf{C}$. Since $\mathbf{C}$ is a Banach space there exists a limit $y_t\in \mathbf{C}$ such that $x_n(t)\longrightarrow y_t$ for all $t\in T$. It is easy to prove that the set $\{y_t|t\in T\}$ is bounded. Hence, the mapping

$$x:T\longrightarrow \mathbf{C}$$

defined by $x(t)=y_t$ for all $t\in T$ is bounded. It follows that $x\in B(T)$ and $x_n\longrightarrow x$ in $B(T)$, because that

$$|x(t)-x_n(t)|=|y_t-x_n(t)|<\varepsilon$$

implies $\|x-x_n\|_\infty<\varepsilon$.

In particular, $\|1\|_\infty=1$ ($1:\ T\longrightarrow \mathbf{C}\ s.t. 1(t)=1$), and thus $B(T)$ is a commutative Banach algebra with unity, because that $\|xy\|_\infty\leq \|x\|_\infty\|y\|_\infty$.

Next, define for each $x\in B(T)$

$$x^*:\ T\longrightarrow \mathbf{C}$$

by $x^*(t)=\overline{x(t)}$ (the conjugate of $x(t)$), then it is easy to prove that for $x,y\in B(T)$ and $\lambda\in \mathbf{C}$

$$x^{**}=x, \qquad (x+y)^*=x^*+y^*,$$
$$(\lambda x)^*=\bar{\lambda}x^*, \qquad (xy)^*=y^*x^*=x^*y^*,$$
$$\|xx^*\|_\infty=\|x\|^2=\|x^*\|^2.$$

In consequence, $B(T)$ is a commutative C^*-algebra with unity. ///

(ii) Let T be a topological space. Then, by (i)

$$B(T)=\{x\colon T\longrightarrow \boldsymbol{C}\mid x \text{ is bounded}\}$$

is a commutative C^*-algebra with unity. Moreover,

$$C^\infty(T)=\{x\in B(T)\mid x \text{ is continuous}\}$$

is a closed C^*-subalgebra of $B(T)$ containing unity. If T is completely regular, then $C^\infty(T)$ separates points of T.

Proof. Suppose a Cauchy sequence $\{x_n\}$ in $C^\infty(T)$. *By*(i), since $B(T)$ is a Banach space there exists a limit $x\in B(T)$ such that $x_n\longrightarrow x$.
Noting that if x_n is continuous and bounded, then $x_n\longrightarrow x$ means that x_n converges uniformly to x, and thus x is also a bounded continuous function, it follows that $x\in C^\infty(T)$. Therefore, $C^\infty(T)$ is a closed subset of $B(T)$. By the same way as in the proof of (i), it is easy to prove that $C^\infty(T)$ is a commutative C^*-algebra with unity.
Next, assume that T is a completely regular space. Each single point of T is a closed subset of T. For two different points $t_1\neq t_2$ in T, there is a continuous function

$$x\colon T\longrightarrow \boldsymbol{R}\subset \boldsymbol{C}$$

such that $x(t_1)=0$, $x(t_2)=1$ and $0\leq x(t)\leq 1$, by the definition of completely regularity, where $\boldsymbol{R}$ is the set of all real numbers. Hence $x\in C^\infty(T)$, and $C^\infty(T)$ separates points of T. ///

(iii) Let T be a locally compact space. Let

$C^\circ(T)=\{x\in C^\infty(T)\mid x$ vanishes at infinity i.e., $\forall\varepsilon>0$ $\{t\in T\mid x(t)\geq\varepsilon\}$ is compact$\}$.

Then $C^\circ(T)$ is a commutative closed C^*-subalgebra. If T is locally compact and Hausdorff, then $C^\circ(T)$ separates points of T.
Moreover, $C^\circ(T)$ has a unity if and only if T is compact.

Proof. At first, we note that

$x\in C^\circ(T)\Longleftrightarrow$ supp $x=$the closure of the set $\{t\in T\mid x(t)\neq 0\}$ is compact.

Also we know that each $x\in C^\circ(T)$ is a bounded continuous function, and $C^\circ(T)\subset C^\infty(T)$.
For a Cauchy sequence $\{x_n\}$ in $C^\circ(T)$ there exists $x\in C^\infty(T)$ such that $x_n\longrightarrow x$ by (ii).

We want to show that $x \in C^{\circ}(T)$, we put

$$L = \{t \in T \mid |x(t)| \geq \varepsilon\}.$$

Since $x_n \longrightarrow x$, there is a N such that $\|x_N - x\|_\infty < \varepsilon/2$. Also, we know that $\{t \in T \mid |x_N(t)| \geq \varepsilon/2\}$ is compact. Since L is a closed subset of $\{t \in T \mid |x_N(t)| \geq \varepsilon/2\}$, L is compact. Therefore, $x \in C^{\circ}(T)$. Hence it is obvious that $C^{\circ}(T)$ is a closed Banach subalgebra of $C^{\infty}(T)$. For each $x \in C^{\circ}(T)$, since x^*: $T \rightarrow \boldsymbol{C}$ is defined by

$$x^*(t) = \overline{x(t)} \quad (t \in T)$$

it follows that $x^* \in C^{\circ}(T)$, and therefore $C^{\circ}(T)$ is a closed C^*-subalgebra of $C^{\infty}(T)$. It is clear that $C^{\circ}(T)$ is commutative.

If T is locally compact and Hausdorff then for $t_1 \neq t_2$ in T, there exists a continuous function

$$x:\ T \longrightarrow \boldsymbol{C}$$

such that $x(t_1) = 0$, $x(t_2) = 1$ and $0 \leq x(t) \leq 1$ for all $t \in T$ by the Uryshon lemma. Moreover, by the Uryshon lemma, we may assume that supp $x = \overline{\{t \in T \mid x(t) \neq 0\}}$ is compact. Hence $x \in C^{\circ}(T)$, and $C^{\circ}(T)$ separates points of T.

It is obvious that

$1 \in C^{\circ}(T)$ $(1:\ T \longrightarrow \boldsymbol{C}:\ 1(t) = 1) \Longleftrightarrow T$ is compact,

and thus our last statement in (iii) has been proved. ///

3. C^*-Algebras Without Unity

Let A be a commutative C^*-algebra with unity. The Gelfand-Naimark theorem ([12]) says that there is an isometric, onto $*$-isomorphism

$$\varphi:\ A \longrightarrow C(\mathscr{M}) \quad (x \longmapsto \hat{x}) \quad \cdots\cdots\cdots\cdots (3\text{—}1),$$

where $\mathscr{M}$ is the maximal ideal space of A, $C(\mathscr{M})$ the commutative C^*-algebra consisting of all continuous functions from $\mathscr{M}$ to $\boldsymbol{C}$ and $\hat{x}$ the *Gelfand transform* of x. In this section, we prove some properties about C^*-algebras without unity. Throughout this section by a C^*-algebra we mean a commutative C^*-algebra

Proposition 3.1. Let A and B be C^*-algebras, and let φ: $A \longrightarrow B$ be a $*$-homomorphism. Then the followings hold:

(i) For all $a \in A$, $\|\varphi(a)\| \leq \|a\|$ and φ is continuous.

(ii) $\varphi(A)$ is a closed C^*-subalgebra of B.

Proof. (i) Let us put

A_I=a C^*-algebra unitification of A

B_I=a C^*-algebra unitification of B (see 4° in section I)

and let e_{A_I} and e_{B_I} identities of A_I and B_I respectively.

Define

$$\tilde{\varphi}: A_I \longrightarrow B_I \quad \cdots\cdots (3\text{—}2)$$

by $\tilde{\varphi}((x, \lambda)) = (\varphi(x), \lambda)$ for $x \in A$ and $\lambda \in C$. Then $\tilde{\varphi}$ is an extension of φ to A_I, i.e., $\tilde{\varphi}$ is a *-homomorphism with $\tilde{\varphi}(e_{A_I}) = e_{B_I}$.

Since $\tilde{\varphi}(x) = \varphi(x)$ for all $x \in A$, by Proposition 1.1 and 4° in section 1,

$$\|\varphi(a)\| = |\tilde{\varphi}(a)| \leq |a| = \|a\|.$$

Hence φ is linear and bounded. That is, φ is continuous.

(ii) Ker $\tilde{\varphi}$ is a closed two side ideal of A_I by Proposition 1.1. Therefore A_I/Ker $\tilde{\varphi}$ is a C^*-algebra with respect to the quotient norm.

Define

$$\bar{\varphi}: A_I/\text{Ker } \tilde{\varphi} \longrightarrow \tilde{\varphi}(A_I) \subset B_I$$

by $\bar{\varphi}([x]) = \tilde{\varphi}(x)$ for all $x \in A_I$. Then $\bar{\varphi}$ is a injective *-homomorplism. By 5° in section 1, $\tilde{\varphi}(A_I)$ is a closed C^*-algebra of B_I.

Hence

$$\varphi(A) = \tilde{\varphi}(A_I) \cap B$$

is a closed C^*-subalgebra of B. ///

Let A be a commutative Banach algebra without unity.

An ideal I of A is said to be *modular* if there exists an element $u \in A$ such that for all $a \in A$

$$a - au \in \text{I}.$$

An element $a \in A$ is said to be *quasi-regular* if there exists an element $b \in A$ such that

$$a \circ b = a + b - ab = 0.$$

For each $x \in A$ we define

$$\sigma_A(x)=\{0\}\cup\{\lambda\in \boldsymbol{C}\mid\lambda\neq 0 \text{ and } \lambda^{-1}x \text{ is not quasi-regular}\}$$

which is called the *spectrum* of x in A. Suppose that A is a commutative Banach algebra without unity, and assume that $f\colon A\longrightarrow \boldsymbol{C}$ is an algebra epimorphism. Then by Proposition 3.1 f is continuous and $\|f\|\leq 1$.

Put

$$M=\{x\in A\mid f(x)=0\},$$

then M is an modular ideal of A, since f is an epimorphism there exists an element $e\in A$ such that $f(e)=1$, That is, for all $x\in A$

$$f(x-x\cdot e)=f(x)-f(x)\ f(e)=0.$$

and thus $x-xe\in M$.

In particular, M is a closed modular maximal ideal of A. In fact, we assume that M is not maximal, then there is a modular maximal ideal N such that $M\subset N$. For each element $a\in N-M$ there exists an element $b\in A$ such that $ab+M=e+M$, because that

$$f_M\colon A/M\longrightarrow \boldsymbol{C}$$

defined by $f_M(a+M)=f(a)$ is a ring isomorphism. Hence $e\in N$ and we have $N=A$.

It follows that M is maximal. Since f is continuous and $M=f^{-1}(0)$, M is a closed ideal of A.

In consequence, for each algebra epimorphism $f\colon A\longrightarrow \boldsymbol{C}$, we see that

(i) f is continuous with $\|f\|\leq 1$.

(ii) the kernel M of f is a modular maximal ideal of A ……(3—3)

(note that each maximal ideal is closed).

Conversely, we can prove that for each modular maximal ideal M of A there is an algebra epimorphism $f_M\colon A\longrightarrow \boldsymbol{C}$ such that $\ker f_M=M$. That is, if we put

$\mathscr{M}=$the set of all modular maximal ideals of A

$\mathscr{X}=$the set of all algebra epimorphisms from A to $\boldsymbol{C}$,

then there is a bijective mapping:

$$\mathscr{M}\longleftrightarrow\mathscr{X}\quad (M\longleftrightarrow f_M). \quad\cdots\cdots(3\text{—}4)$$

Lemma 3.2. With the above notations the followings hold.

(i) $\mathscr{X}\cup\{0\}=\{f\colon A\longrightarrow \boldsymbol{C}\mid f$ is a continuous linear form such that $f(xy)=f(x)f(y)\}$

is a weak *-compact subset of A', where A is a Banach algebra without unity and A' the dual space of A.
Therefore $\mathscr{X}$ is locally compact for the weak *-topology. The set $\mathscr{M}$, equipped with the topology induced by $\mathscr{X}$, is called the *modular maximal ideal space* of A.

(ii) For each $x \in A$

$$\hat{x}: \mathscr{M} \longrightarrow \boldsymbol{C} \ (\hat{x}(M) = f_M(x))$$

is continuous and vanishes at infinity, i.e., $\hat{x} \in C^\circ(\mathscr{M})$ (see (iii) of example 2.1 in §2.). In particular,

$$\sigma_A(x) = \hat{x}(\mathscr{M}) \cup \{0\}, \qquad \|\hat{x}\|_\infty = r_A(x) \leq \|x\|,$$

where $r_A(x) = \sup\{|\lambda| \ |\lambda \in \sigma_A(x)\}$.

(iii) $A \longrightarrow C^\circ(\mathscr{M})$ $(x \longmapsto \hat{x})$ is an algebra homomorphism.

The kernel of this homomorphism is the radical of A, where the radical of A is the intersection of all modular maximal ideals of A.

Proof. (i) At first, we shall prove that if $f: A \longrightarrow \boldsymbol{C}$ is a continuous and linear form such that $f(xy) = f(x)\, f(y)$ and $f \not\equiv 0$, then f is an algebra epimorphism. There exists at least one element $a \in A$ such that $f(a) \neq 0$. In this case,

$$f\,|\,Ca (Ca \subset A): \ Ca \longrightarrow \boldsymbol{C}$$

is an abelian group isomorphism, and thus $f: A \longrightarrow \boldsymbol{C}$ is an algebra epimorphism.
By (3—3) and (3—4) $\mathscr{X} \cup \{0\}$ is contained in the closed unit ball of A' which is weak *-compact by the Alaoglu-Bourbaki theorem([1]). Hence it will suffices to show that $\mathscr{X} \cup \{0\}$ is weak *-closed in A'. Suppose $\{f_j\}$ is a net in $\{0\} \cup \mathscr{X}$ which is weak *-convergent to $f \in A'$ (i.e., $\forall x \in A \ \lim_{n\to\infty} |f_n(x) - f(x)| = 0$). Then for x, $y \in A$ and $\lambda \in \boldsymbol{C}$,

$$\lim f_j(x+y) = \lim f_j(x) + \lim f_j(y) \Longrightarrow f(x+y) = f(x) + f(y)$$
$$\lim f_j(xy) = \lim f_j(x) f_j(y) = \lim f_j(x) \ \lim f_j(y) \Longrightarrow f(xy) = f(x)\, f(y)$$
$$\lim f_j(\lambda x) = \lambda \lim f_j(x) \Longrightarrow f(\lambda x) = \lambda f(x),$$

and thus $f \in \mathscr{X} \cup \{0\}$. Since $\mathscr{X} \cup \{0\}$ is Hausdorff and closed in a compact set it is obvious that $\mathscr{X} \cup \{0\}$ is a weak *-compact subset of A'.
$\{0\}$ is a closed subset of A', and thus $\mathscr{X} = \mathscr{X} \cup \{0\} - \{0\}$ is a locally compact subset for the weak *-topololgy.

(ii) Put

$$L = \{M \in \mathscr{M} \mid |\hat{x}(M)| = |f_M(x)| \geq \varepsilon\}$$

for some positive number $\varepsilon>0$. Let $\{M_j\}$ be a net in L which is weak $*$-convergent to $M\in\mathcal{M}$. If $|\hat{x}(M)|=\varepsilon_1<\varepsilon$, then for $0<\delta<\varepsilon-\varepsilon_1$

$$|\hat{x}(M_j)-\hat{x}(M)|>\delta.$$

Hence we have $|\hat{x}(M)|\geq\varepsilon$ and $M\in L$. That is, L is a closed subset of $\mathcal{M}$ and thus L is compact. Hence, it follows that for all $x\in A$, $\hat{x}\in C^\circ(\mathcal{M})$.

For each $\lambda\in\sigma_A(x)$ let us put

$$L(\lambda)=\text{the ideal of } A \text{ generated by } \{y-\lambda^{-1}xy\,|\,y\in A\}.$$

Then $\lambda^{-1}x\notin L(\lambda)$, because that if there exist elements $y,z\in A$ such that $z(y-\lambda^{-1}xy)=-\lambda^{-1}x$ then $(zy)+\lambda^{-1}x-\lambda^{-1}x(zy)=0$, i.e., $\lambda\notin\sigma_A(x)$. Therefore $L(\lambda)$ is a proper ideal of A. Let $M(\lambda)$ be a maximal ideal containing $L(\lambda)$. Hence $M(\lambda)$ is a modular maximal ideal of A. Since

$$f_{M(\lambda)}(\lambda^{-1}x-\lambda^{-2}x^2)=0,$$

we have $\hat{x}(M(\lambda))=\lambda$. This shows that $\sigma_A(x)$ is contained in $\hat{x}(\mathcal{M})\cup\{0\}$.

Conversely, we assume that $\hat{x}(M)=\lambda$, where M is a modular maximal ideal of A $(\lambda\neq0)$.
Then

$$f_M(x)=\lambda\Longrightarrow f_M(\lambda^{-1}x)=1\Longrightarrow \forall a\in A \quad f_M(a-\lambda^{-1}xa)=0,$$

hence $a-\lambda^{-1}xa\in M$ and $\lambda^{-1}x$ is the unity element mod M.
Suppose that for some element $y\in A$

$$y+\lambda^{-1}x-\lambda^{-1}xy=0.$$

Then $y-\lambda^{-1}xy=-\lambda^{-1}x\in M$ which contradicts to $f_M(\lambda^{-1}x)=1$.
That is, for all $y\in A$ $y+\lambda^{-1}x-\lambda^{-1}xy\neq0$, and thus $\lambda\in\sigma_A(x)$.
In consequence, we have

$$\sigma_A(x)=\{\hat{x}(M)\,|\,M\in\mathcal{M}\}\cup\{0\}.$$

Therefore

$$\|\hat{x}\|_\infty=\sup_{M\in\mathcal{M}}|\hat{x}(M)|=r_A(x).$$

Since

$$|\hat{x}(M)|=|f_M(x)|\leq\|x\|$$

by (3—3) it is obvious that $\|\hat{x}\|_\infty = r_A(x) \leq \|x\|$.

(iii) For all $x, y \in A$, $\lambda \in C$ and $M \in \mathscr{M}$

$$(x+y)^\wedge(M) = f_M(x+y) = f_M(x) + f_M(y) = \hat{x}(M) + \hat{y}(M),$$
$$(\lambda x)^\wedge(M) = f_M(\lambda x) = \lambda f_M(x) = \lambda \hat{x}(M),$$
$$(xy)^\wedge(M) = f_M(xy) = f_M(x) f_M(y) = \hat{x}(M)\ \hat{y}(M),$$

and thus the mapping

$$\psi:\ A \longrightarrow C^\circ(\mathscr{M})\ \ (x \longmapsto \hat{x})$$

(which is the Gelfand transformation) is an algebra homomorphism (see(iii) of Example 2.1 in section 2). In particular, the kernel of ψ is the set $\bigcap_{M \in \mathscr{M}} M$. ///

Let A be a commutative C^*-algebra without unity, and let A_1 be a C^*-algebra unitification of A (see 4° in section 1).

We shall put

$\mathscr{M}$ = the modular maximal ideal space of A.

$\mathscr{M}_1$ = the maximal ideal space of A_1.

Lemma 3.3. There is a homeomorphism

$$\psi:\ \mathscr{M}_1 - \{A\} \longrightarrow \mathscr{M}$$

Proof. For each $\tilde{M} \in \mathscr{M}_1 - \{A\}$ there is a unique algebra epimorphism $f_{\tilde{M}}: A_1 \longrightarrow C$ such that ker $f_{\tilde{M}} = \tilde{M}$. Since $A \not\subset \tilde{M}$, there exists an element $x \in A - \tilde{M}$ such that $f_{\tilde{M}}(x) = \lambda \neq 0$. Hence we have $f_{\tilde{M}}(\lambda^{-1}x) = 1$.

Put

$$M = A \cap \tilde{M}$$

then it is easy to see that $\{a - \lambda^{-1}xa \mid a \in A\} \subset M$. In particular, M is a modular maximal ideal of A. In fact, it is clear that $\lambda^{-1}x$ is a unity mod M.

Assume that M is not maximal in A, then there exists a proper ideal $\bar{M}$ of A such that $M \subsetneqq \bar{M} \subsetneqq A$.

Put

L = the ideal of A_1 generated by $\bar{M} \cup \tilde{M}$

Then, it is a proper ideal of A_1, because that if $L=A_1$ then there exist two elements $a\in\bar{M}$ and $x\in\tilde{M}$ such that

$$a+x=e_{A_1},$$

and that

$$x=e_{A_1}-a\in\tilde{M}\Longrightarrow f_{\tilde{M}}(a)=1\Longrightarrow{}^{\forall}b\in A\ b-ab\in\bar{M}$$
$$\Longrightarrow b\in\bar{M}((\because)ab\in\bar{M})\Longrightarrow A=\bar{M}.$$

Therefore, we have a contradiction to the maximality of $\tilde{M}$, and M must be maximal in A. Now, we define

$$\phi:\ \mathscr{M}_1-\{A\}\longrightarrow\mathscr{M}$$

by $\phi(\tilde{M})=M$. Then inverse ϕ^{-1} of ϕ is defined as follows.
For each $M\in\mathscr{M}$ and the algebra epimorphism f_M: $A\longrightarrow \boldsymbol{C}$, we define $f_{\tilde{M}}$: $A_1\longrightarrow \boldsymbol{C}$ by

$$f_{\tilde{M}}(\lambda e_{A_1}+x)=\lambda+f_{\tilde{M}}(x).$$

where $\lambda(\neq 0)\in\boldsymbol{C}$ and $x\in A$. Then $f_{\tilde{M}}$ is an algebra epimorphism. We put $\tilde{M}=\ker f_{\tilde{M}}$ and define $\phi^{-1}(M)=\tilde{M}$. It follows that

$$\phi^{-1}\phi=1_{\mathscr{M}_1-\{A\}},\quad \phi\phi^{-1}=1_{\mathscr{M}}.$$

As is well-known, $\mathscr{M}_1$ is a compact Hausdorff space for the weak*-topology (or the Gelfand topology). Therefore $\mathscr{M}_1-\{A\}$ is a locally compact space, and by (i) of Lemma 3.2 $\mathscr{M}$ is also a locally compact space for the weak*-topology.
Next, we want to prove that ϕ and ϕ^{-1} are continuous to the weak*-topologies. For each $\tilde{M}\in\mathscr{M}_1-\{A\}$ let $U(M)$ be an open neighborhood of $\phi(\tilde{M})=M$ in $\mathscr{M}$. Then there are $x_1, x_2,\cdots, x_n$ of A and $U_1, U_2\cdots U_n$ such that

$$\hat{x}_1^{-1}(U_1)\cap\cdots\cap\hat{x}_n^{-1}(U_n)\cap\mathscr{M}=U(M),$$

where U_i $(i=1,2\cdots,n)$ is an open neighborhood of $\hat{x}_i(M)$ in $\boldsymbol{C}$.
is easy to see that Noting that $f_M=f_{\tilde{M}}|A$ it

$$\phi^{-1}(U(M))=\hat{x}_1^{-1}(U_1)\cdots\cap\hat{x}_n^{-1}(U_n)\cap\{\mathscr{M}_1-\{A\}\},$$

which is an open subset of $\mathscr{M}_1-\{A\}$. Hence ϕ is continuous.
Similary, we can prove that ϕ^{-1} is continuous. ///
Note that $\mathscr{M}_1$ and $\mathscr{M}\cup\{A\}$ are homeomorphic.

Theorem 3.4. Let A be a commutative C^*-algebra without unity, and let $\mathcal{M}$ be the modular maximal ideal space of A. Then, the Gelfand transformation maps A isometrically and *-isomorphically onto $C^\circ(\mathcal{M})$, where $C^\circ(\mathcal{M})$ is the commutative C^*-algebra consisting of all bounded continuous functions vanishing at infinity from $\mathcal{M}$ to $\boldsymbol{C}$ (see(ii) of Lemma 3.2).

Proof. Let A_I be a C^*-algebra unitification of A, and let $\mathcal{M}_I$ be the maximal ideal space of A_I. Since every element x of A_I is normal we have $r_{A_I}(x)=|x|$ by 2° in section 1. By 4° in section 1 for $x\in A$ $\|x\|=|x|$, and thus $r_A(x)=\|x\|$. Hence, by (ii) of Lemma 3.2 we have $\|\hat{x}\|_\infty=r_A(x)$ for all $x\in A$, and thus $\|\hat{x}\|_\infty=\|x\|$. Therefore, by (ii) and (iii) of Lemma 3.2 the the mapping

$$\eta:\ A\longrightarrow C^\circ(\mathcal{M})\ \ (x\longmapsto\hat{x})$$

is an isometric algebra homomorphism.
Recall the mapping (3—1)

$$\varphi:\ A_I\longrightarrow C(\mathcal{M}_I)\ \ (x\longmapsto\hat{x}),$$

which is isometric, *-isomorphic and onto. Noting that $f_M=f_{\tilde{M}}|A$ where $\psi(\tilde{M})=M$ (for notations see Lemma 3.3) and $\varphi(x^*)=\varphi(x)^*$ (i.e., $\hat{x^*}=\hat{x}^*$) we see that η is a *-monomorphism ($\forall x\in A\ \varphi(x)|\mathcal{M}=\eta(x)$). We put

$$C^\circ(\mathcal{M}_I)=\left\{f\in C(\mathcal{M}_I)\ \middle|\ \begin{array}{l}① \ f|A\equiv 0\\ ② \ \forall\varepsilon>0\ \{\tilde{M}\in\mathcal{M}_I\,|\,|f(\tilde{M})|\geq\varepsilon\}\text{ is compact.}\end{array}\right\}$$

Then, by the isomorphism φ in (3—1) there exists only one element $x+\lambda e_{A_I}\in A_I(x\in A, \lambda\in C)$ such that $(x+\lambda e)\hat{}=f$ for each $f\in C^\circ(\mathcal{M}_I)$.

But, $f(A)\equiv 0$ implies that $\lambda=0$ because that $\hat{x}(A)\equiv 0$ and $\hat{e}_{A_I}\ (A)=1$.
Therefore, we have

$$A\cong\varphi(A)=\{\hat{x}\,|\,x\in A\}=C^\circ(\mathcal{M}_I).$$

By (ii) of Lemma 3.2 (or by Lemma 3.3), we see that $\mathcal{M}\cup\{A\}$ is the Alexandroff one-point compactification of $\mathcal{M}$ ([11]). Hence each bounded continuous function $g\in C^\circ(\mathcal{M})$ can be extended to a bounded continuous function defined on $\mathcal{M}\cup\{A\}$ setting $g(\{A\})=0$.
Hence we have

$$C^\circ(\mathcal{M})=C^\circ(\mathcal{M}\cup\{A\}).$$

Let ψ: $\mathscr{M}_1 \rightarrow \mathscr{M} \cup \{A\}$ be the homeomorphism in Lemma 3.3.

Define

$$\tau:\ C^{\circ}(\mathscr{M}_1) \longrightarrow C^{\circ}(\mathscr{M} \cup \{A\})$$

by $\tau(f) = f \circ \psi^{-1}$ for each $f \in C^{\circ}(\mathscr{M}_1)$, Then $\tau^{-1}(g) = g \circ \psi$ for each $g \in C^{\circ}(\mathscr{M} \cup \{A\})$, which is the inverse of τ.

It follows that τ is an algebra isomorphism.

Hence we have isomorphisms

$$A \cong C^{\circ}(\mathscr{M}_1) \cong C^{\circ}(\mathscr{M} \cup \{A\}) = C^{\circ}(\mathscr{M}).$$

In consequence

$$\eta:\ A \longrightarrow C^{\circ}(\mathscr{M}) \quad (x \longmapsto \hat{x})$$

is isometric, *-isomorphic and onto. ///

4. States and *-Representations

For a Hilbert space H the set $B(H)$ of all linear bounded operators from H to itself is a C^*-algebra as follows.

As is well-known, for each $T \in B(H)$ its norm is defined by

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \quad (x \in H).$$

For each $T \in B(H)$ T^* is defined by

$$(Tx, y) = (x, T^*y) \quad (x, y \in H),$$

where (,) is the given inner product in H ([12]).

For each $T \in B(H)$ the set

$$W(T) = \{(Tx, x) \mid x \in H,\ \|x\| = 1\}$$

is called the ***numerical range*** of T.

Under our situation the following are known ([1])

$$\left.\begin{array}{l} \text{(i) } W(T) \text{ is a convex subset of } \boldsymbol{C} \\ \text{(ii) } \sigma(T) \subset \overline{W(T)} \\ \text{(iii) If } T \text{ is normal } (TT^* = T^*T) \text{ then conv } \sigma(T) = \overline{W(T)}, \end{array}\right\} \cdots\cdots (4-1)$$

where conv $\sigma(T)$ is the convex hull of $\sigma(T)$ and $\overline{W(T)}$ is the closure in $\boldsymbol{C}$ of $W(T)$.

Definition 4.1. Let A be a C^*-algebra with unity and let H be a Hilbert space. A *-homomorphism

$$\varphi: A \longrightarrow B(H)$$

is called a **-representation* of A on H.

If $\varphi(e_A)=1_H$ (e_A is the unity of A and 1_H: $H \longrightarrow H$ is the identity map) then φ is said to be *unital*.

If φ is a *-monomorphism (injective), then φ is called a *faithful *-representation* of A on H.

Definition 4.2. Let A be a C^*-algebra, and let f: $A \longrightarrow \boldsymbol{C}$ be a linear form. If for all $a \in A$

$$f(a^*a) \geq 0$$

then f is called a *state* defined on A.

Let f: $A \longrightarrow \boldsymbol{C}$ be a state on A.

If A has a unity e and $f(e)=1$, then f is said to be *normalized.*

In particular, for any element $a \in A$ there exists a normalized state f such that $f(a^*a) = \|a^*a\|$.

To prove the Gelfand-Naimark representation theorem ([1]), for a C^*-algebra A with unity it is general to construct a unital *-representation φ: $A \longrightarrow B(H)$ and a vector $u \in H$ for a given state f on A such that

$$f(a)=(\varphi(a)u, u) \quad \cdots\cdots (4-2)$$

for all $a \in A$, where (,) is the given inner product in H.

In this case, H, (,) and u are defined as follows.

At first, define an inner product $<$, $>$ in A by

$$<x, y> = f(y^*x).$$

Next, we put

$$N = \{x \in A \mid \forall y \in A \ <x, y> = 0\}$$

and

$$E = A/N$$

Define an inner product (,) in E by

$$(\tilde{x}, \tilde{y}) = \langle x, y \rangle,$$

where for the canonical map η: $A \longrightarrow A/N = E$ $\tilde{x} = \eta(x)$.
It is easy to prove that (,) is an inner product. Thus E is a pre-Hilbert space.
Let H be the completion of E. That is, we regard E as a dense linear subspace of the Hilbert space H. Define

$$\varphi: A \longrightarrow B(H) \quad \cdots\cdots\cdots (4\text{—}3)$$

by $\varphi(a)\tilde{x} = (ax)^{\sim}$ for all $a \in A$ and $\tilde{x} \in H$.
In particular,

$$u = \tilde{e}_A = e_A + N,$$

where e_A is the unity of A.

Definition 4.3. Let φ: $A \longrightarrow B(H)$ be a *-representation of A on H.
If there exists an elements $u \in H$ such that

$$\{\varphi(a)u \mid a \in A\}$$

is a dense subset of H then φ is said to be *cyclic* and u is called a *cyclic vector* for the representation φ.

Corollary 4.4. The unital *-representation φ: $A \longrightarrow B(H)$ in (4-3) is cyclic.

Proof. It is clear that u is a cyclic vector of the representation φ because that

$$\{\varphi(a)u \mid a \in A\} = E$$

and $\bar{E} = H$. ///

Lamma 4.5. Let φ: $A \longrightarrow B(H)$ and ψ: $A \longrightarrow B(K)$ be cyclic *-representations with cyclic vectors u and v, respectively.
If for all $a \in A$

$$(\varphi(a)u, u) = (\varphi(a)v, v)$$

then there is a unitary operator W: $H \longrightarrow K$ which is isometric, where if $(\varphi(a)u, u) = 0$, $\varphi(a)u = 0$

Proof. We put

$$H_o = \{\varphi(a)u \mid a \in A\} \text{ and } K_o = \{\varphi(a)v \mid a \in A\}$$

Then, our hypothesis $\bar{H}_o = H$ and $\bar{K}_o = K$ we define

$$W_o: H_o \longrightarrow K_o$$

by $W_o(u)=v$ and $W_o(\varphi(a)u)=\psi(a)v$ for all $a\in A$. Then W_o is bijective and linear, In fact,

$$W_o(\varphi(a)u)=\psi(a)v=0 \Longrightarrow \varphi(a)u=0$$

because that $(\varphi(a)u,u)=(\psi(a)v,v)=0$, and

$$W_o((\varphi(a_1)+\varphi(a_2))u)=W_o(\psi(a_1+a_2)u)=\psi(a_1+a_2)v$$
$$=\psi(a_1)v+\psi(a_2)v=W_o(\varphi(a_1)u)+W_o(\varphi(a_2)u)).$$

Next, we define W_o^* by

$$(W_o(\varphi(a)u),\ \psi(b)v)=(\varphi(a)u,\ W_o^*(\psi(b)v)),$$

where $a,b\in A$. Then W_o^*: $K_o \longrightarrow H_o$ is also linear and bijective. Since

$$(W_o(\varphi(a)u),\ \psi(b)v)=(\psi(a)v,\ \psi(b)v)=(\psi(b^*a)v,v)$$
$$=(\varphi(b^*a)u,u)=(\varphi(a)u,\ \varphi(b)u)$$

we have $W_o^*(\psi(b)v)=\varphi(b)u$. Therefore

$$W_o^*W_o=1_{Ho},\quad W_oW_o^*=1_{ko},$$

and thus W_o is a unitary operator. Moreover, since

$$||u||^2=(\varphi(e_A)u,u)=(\psi(e_A)v,v)=||v||^2,$$
$$||\varphi(a)u||^2=(\varphi(a)u,\ \varphi(a)u))=(\varphi(a^*a)u,u)=(\psi(a^*a)v,v)$$
$$=(\psi(a)v,\ \psi(a)v)=||\psi(a)v||^2$$

W is an isometric isomorphism. ///

Thereom 4.6. Let φ: $A\longrightarrow B(H)$ and ψ: $A\longrightarrow B(K)$ be *-representations of A which are induced from a given state f on A (see(4-3)).
Then φ and ψ are unitary equivalent (i.e., there exists a unitary operator W: $H\longrightarrow K$, which is isometric).

Proof. By corollary 4.4 the representations φ and ψ are cyclic.
We put

$$u=\text{a cyclic vector of } \varphi,\quad v=\text{a cyclic vector of } \psi,$$

then by (4-2) we have

$$f(a)=(\varphi(a)u,u)=(\psi(a)v,v)$$

for all $a \in A$. Therefore, by Lemma 4.5 we have a unitary operator $W: H \longrightarrow K$, which is isometric. ///

Definition 4.7. Let A be a C^*-algebra with unity e_A.
We put

Σ = the set of all normalized states on A.

For each $a \in A$ we write

$$\Sigma(a) = \Sigma_A(a) = \{f(a) \mid f \in \Sigma\}$$

which is called the *numerical status* of a in A.
Let A be a C^*-algebra with unity, and let B be a closed C^*-subalgebra of A containing e_A.
For a linear form $f: A \longrightarrow \boldsymbol{C}$, since

$$f \text{ is a state} \Longleftrightarrow f \text{ is continuous and } \|f\| = f(e_A) \quad ([1]),$$

by using the Hahn-Banach theorem ([11]) we can prove that

$$\Sigma_B(b) = \Sigma_A(b) \quad \cdots\cdots (4\text{-}4)$$

for all $b \in B$. Futhermore, if $a \in A$ is hermitian ($a = a^*$) we can prove that

$$\Sigma(a) = \mathrm{conv}(a) \quad \cdots\cdots (4\text{-}5) \quad ([1]).$$

Theorem 4.8. Let A be a C^*-algebra with unity e_A.
Then the followings hold.

(i) For a C^*-algebra B with unity e_B, if $\varphi: A \longrightarrow B$ is a $*$-monomorphism such that $\varphi(e_A) = e_B$, then for all $a \in A$

$$\Sigma_A(a) = \Sigma_B(\varphi(a)).$$

(ii) For a Hilbert space H if $\varphi: A \longrightarrow B(H)$ is a faithful unital $*$-representation of A on H then for all $a \in A$

$$\Sigma_A(a^*a) = \overline{W(\varphi(a^*a))} \quad (\text{see}(4\text{-}1))$$

Proof. (i) By 5° and (ii) of Proposition 3.1 $\varphi: A \cong \varphi(A)$ is an isometric isomorphism and $\varphi(A)$ is a closed C^*-subalgebra of B containing e_B. Hence by (4-4)

$$\Sigma_{\varphi(A)}(\varphi(a)) = \Sigma_B(\varphi(a)).$$

An one-to-one corresponding

$$\Sigma_A \longleftrightarrow \Sigma_{\varphi(A)}$$

is defined by $f=\tilde{f}\circ\varphi$ for $\tilde{f}\in\Sigma\varphi(A)$. By Proposition 3.1, since φ is continuous, $f=\tilde{f}\circ\varphi\in\Sigma_A$.

Conversely, for each $f\in\Sigma_A$ $\tilde{f}=f\circ\varphi^{-1}$ is a state on $\varphi(A)$. It follows that for all $a\in A$

$$\Sigma_A(a)=\Sigma_{\varphi(a)}(\varphi(a))=\Sigma_B(\varphi(a)).$$

(ii) For all $a\in A$ it is important to see that a^*a and $\varphi(a^*a)$ are hermitian elements in A and in $B(H)$, respectively. Since each hermitian element is normal. by (4-1) we have

$$\mathrm{conv}\sigma(\varphi(a^*a))=\overline{W(\varphi(a^*a))}$$

On the other hand, by (4-5),

$$\mathrm{conv}\sigma(\varphi(a^*a))=\Sigma_B(\varphi(a^*a)).$$

By (i), $\Sigma_B\,\varphi(a^*a)=\Sigma_A(a^*a)$, and thus

$$\Sigma_A(a^*a)=\overline{W(\varphi(a^*a))}. \quad ///$$

References

1. S.K. Berberian: Lectures in Functional Analysis and Operator Theory, Springer-Verlag, 1974.
2. S.K. Berberian and G.H. Orland: On the closure of the numerical range of an operator, *Proc Amer. Math, Soc.* ***18***, pp.499~503 (1967)
3. J. Kixmier: Les C*-algebras et leurs represtations. second edition, Paris: Gauthier-Villars (1969)
4. R.C. Douglas and C. Pearcy: On the spectral theorem for normal Operators, *Proc, cambridge philos, Soc.* ***68***, 393~400 (1970)
5. R. Godement: Sur la theoric des representations *Unitaries Ann of Math* (2) ***53***, pp.68~124 (1951)
6. H.H Golastine: Weakly complete Banach spaces, *Duke Math J.* ***4***, pp.125~131 (1938)
7. P.R Halmos: Continuous functions of Hermitian Operators, Proc. *Amer. Math. Soc.* ***31***, pp.130~132 (1972)
8. R.V. Kakutani: On the orthogonalization of operators representations. *Amer. J. Math.* ***77***. pp.600~620 (1955)
9. M. A Naimark: Normed Rings, Revised English edition, Translated from the Russian by Leo F. Boron. Groningen: Noordholl (1964)
10. C.E. Rickart: General theory of Banach algebras, Princeton, N.J.: Van Nostrand (1960)
11. H.L. Royden: Real Analysis, Macmillan Company, London (1968)
12. W. Rudin: Functional Analysis, McGraw-Hill, Inc.
13. S. Sakai: C*-algebras and W*-algebras, Springer-Verlag (1971)
14. J.T. Schwartz W*-algebras. Gordon and Breach (1967)
15. J. Wermer: Banach algebras and analytic functions. *Adv. Math.* ***I***. pp.51~102 (1961)
16. K. Yosida: Functional analysis, Springer-Verlag (1965)
17. A.C. Zaanen: Linear analysis. Measure and integral, Banach and Hilbert space, linear integral equations. Amsterdam North-Holland (1953)

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