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THE CLASS OF LIMIT LAWS FOR LÈVY PROCESSES

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1. Introduction

Let X_t be R^1-valued stochastic process with stationary independent increments whose log characteristic function is given by

$$(1.1)\qquad \log E\exp(iuX_t)=t[ibu-2^{-1}\sigma^2u^2 + \int(\exp(iux)-1-iux(1+x^2)^{-1})d\nu(x)]$$

where ν is a Borel measure on $R-\{0\}$ satisfying $\int x^2(1+x^2)^{-1}d\nu(x)<\infty$. As usual, we assume that X_t is normalized so that $X_0=0$ and almost all sample paths are right continuous and possess left limits at every t. Our main concern is related to the following classical problem; the convergence in distribution of $(X_t-B_t)/A_t$ as $t\to 0$ and $t\to\infty$, where $\lim_{t\to 0} A_t=0$ and $\lim_{t\to\infty} A_t=\infty$.

The analogous problem was completely solved for the case of sum of i.i.d. random variables X_n with common distribution function F. It is well-known that the class of limit distributions of normed sum, $(S_n-B_n)/A_n$, coincides with the stable distributions and necessary and sufficient conditions for F to be in the domain of attraction of each stable distribution are expressed in terms of F. As pointed out in [8] by Pruitt, there is duality between the distribution function for the sum of i.i.d. random variables and Lèvy measure for the Lèvy process. Furthermore, the analogous condition to the case of i.i.d. random variables with Lèvy measure replacing distribution function was obtained in [10] by Wee for the convergence to the normal distribution as follows; it is necessary and sufficient for $\{A_t\}$ and $\{B_t\}$ to exist such that $(X_t-B_t)/A_t\xrightarrow{\mathcal{D}}\mathcal{N}(0.1)$ as $t\to 0$ and $t\to\infty$ that

$$\frac{x^2\nu\{y:|y|>x\}}{\sigma^2+\int_{|y|\le x}y^2d\nu(y)}\to 0$$

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as $x \to 0$ and $x \to \infty$ respectively. In fact, our present work is motivated by this paper and suggestion by Professor Puritt· Surprisingly it turns out that similar results to the case of normed sum of i. i. d. random variables again hold with Lèvy measure in place of distribution function. Note that it is obvious that the limit distribution is necessaily stable for $t \to \infty$ but not for $t \to 0$. Before stating our main result, define for $a > 0$,

$$G^{\nu}(a) = \int_{|x|>a} d\nu(x)$$
$$K^{\nu}(a) = a^{-2}(\sigma^2 + \int_{|x|\le a} x^2 d\nu(x))$$
$$G^{\nu}_{+}(a) = \int_{x>a} d\nu(x)$$
$$G^{\nu}_{-}(a) = \int_{x<-a} d\nu(x)$$

On first result is

THEOREM 1. (*a*) (1) *The class of limit distributions of* $\{(X_t - B_t)/A_t\}$ *as* $t \to 0$ *for suitable numbers* $\{A_t\}$, $\{B_t\}$ *coincides with the family of stable distributions.*

(2) *For some* $\{A_t\}$, $\{B_t\}$, *the limit of* $(X_t - B_t)/A_t$ *as* $t \to 0$ *has stable distribution with index* α, $0 < \alpha < 2$ *iff* G *is regularly varying at* 0 *with order* $-\alpha$ *and* $G^{\nu}_{+}(x)/G^{\nu}(x) \to C_1/(C_1 + C_2)$, $G^{\nu}_{-}(x)/G^{\nu}(x) \to C_2/(C_1 + C_2)$ *as* $x \to 0$ *for some constants* $C_1, C_2 \geqslant 0$.

(*b*) *The similar conclusions hold if* $t \to 0$ *and* $x \to 0$ *are replaced by* $t \to \infty$ *and* $x \to \infty$ *respectively.*

Another related topic is stochastic compactness of family $\{(X_t - B_t)/A_t\}$ in appropriate sense. This means that every sequence $\{(X_{t_n} - B_{t_n})/A_{t_n}\}$ with $t_n \to 0$ or $t_n \to \infty$ has further subsequence which converges to nondegenerate distribution. In the case of normed sum of i. i. d. random variables, Feller [2] observed that

$$\lim_{\to\infty} \frac{x^2(1 - F(x) + F(-x))}{\int_{|y|\le x} y^2\, dF(y)} < \infty$$

is equivalent to the stochastic compactness of $\{(S_n - B_n)/A_n\}$ for some sequence $\{A_n\}$ and $\{B_n\}$. Recently very interesting equivalent conditions to stochastic compactness were obtained by Jain & Orey in [6] and Griffin, Jain, and Pruitt in [4]. For more related work to stochastic compactness of normed sum of i. i. d. random variables, see Hall [5].

Now back to Lévy process, we shall see that the analogue of Feller's condition near zero and infinity is equivalent to stochastic compactness of $\{(X_t-B_t)/A_t\}$ in appropriate sense. Define for $a>0$,

$$g(a)=G^{\nu}(a)+K^{\nu}(a).$$

The function g is continuous and strictly decreasing once g reaches its support and $g(a)\to 0$ as $a\to\infty$. We assume that $\int d\nu(x)=\infty$ since then $g(a)\to\infty$ as $a\to 0$. If $\int d\nu(x)<\infty$, then one can easily see that X_t is sum of a compound Poisson process and λt for some constant λ, which is less interesting. Under our present assumption we may define a_t by

$$g(a_t)=1/t.$$

We shall see that for this $\{a_t\}$, there exists $\{b_t\}$ such that $\{(X_t-b_t)/a_t\}$ is stochastically compact in some sense. Our result concerning stochastic compactness is

THEOREM 2. (*a*) *The following three statements are equivalent;*

(1) *There exist* $\{A_t\}$, $\{B_t\}$ *such that every sequence* $\{(X_{t_n}-B_{t_n})/A_{t_n}\}$ *for* $t_n\to 0$ *has further subsequence which converges to nondegenerate distribution.*

(2)
$$\lim_{T\to\infty}\overline{\lim_{t\to 0}}\, tG^{\nu}(Ta_t)=0.$$
$$\underline{\lim_{t\to 0}}\, tK^{\nu}(\xi a_t)>0.$$

for some $\xi>0$.

(3)
$$\lim_{x\to 0}\frac{G^{\nu}(x)}{K^{\nu}(x)}<\infty.$$

(*b*) *Similarly for* $t\to\infty$ *and* $x\to\infty$.

Another interesting question related to stochastic compactness is to describe the class of distributions which can be obtained as the limits of subsequences under the assumption of stochastic compactness. For stochastic compact family of sum of i. i. d. random variables, the class of limit distributions obtained as the limit of subsequences was completely characterized by Pruitt in [7].

2. Limit laws of Lèvy processes.

Recall the expression of the log characteristic function of X_t in (1. 1) and denote it by (b, σ^2, ν). Besides G^{ν}, K^{ν}, G^{ν}_{+} and G^{ν}_{-} defined in

Section 1, we introduce following definitions; for $a>0$

$$\beta^{\nu}(a)=\int_{|x|\le a} x^3/(1+x^2)\,d\nu(x)$$
$$\alpha^{\nu}(a)=\int_{|x|>a} x/(1+x^2)\,d\nu(x).$$

First we need to prove the central convergence criterion in terms of Lèvy measure. Observe that if $(X_t-B_t)/A_t$ converges to Y in distribution as $t\to 0$ or $t\to\infty$, then Y has necessarily infinitely divisible distribution. So we denote Lèvy representation of its characteristic function by $(\bar{b},\bar{\sigma}^2,\bar{\nu})$ and corresponding functions by $G^{\bar{\nu}}, K^{\bar{\nu}}$ and etc.

LEMMA 1. *(a) In order that for some constants $\{B_t\}$, $(X_t-B_t)/A_t$ converges to Y in distribution as $t\to 0$, it is necessary and sufficient that following conditions be satisfied;*

(1) $\lim_{\varepsilon\to 0}\overline{\lim_{t\to 0}}\ \varepsilon^2 tK(\varepsilon A_t)=\lim_{\varepsilon\to 0}\underline{\lim_{t\to 0}}\ \varepsilon^2 tK(\varepsilon A_t)=\bar{\sigma}^2.$

(2) *At every continuity point y of $G^{\bar{\nu}}_+$ and $G^{\bar{\nu}}_-$,*

$$\lim_{t\to 0} tG^{\nu}_+(y\ A_t)=G^{\bar{\nu}}_+(y)$$
$$\lim_{t\to 0} tG^{\nu}_-(y\ A_t)=G^{\bar{\nu}}_-(y).$$

The constants B_t are determined by

$$B_t=t(\beta^{\nu}(\xi A_t)-\alpha^{\nu}(\xi\ A_t)+b)+MA_t$$

where M is an appropriate constant and ξ is continuity point of $\bar{\nu}$.

(b) The similar statements hold if "$t\to 0$" is replaced by "$t\to\infty$".

Proof. (a) First we show that (1) and (2) are sufficient. Consider following log characteristic function of $(X_t-B_t)/A_t$;

(2.1) $\log E\exp[iu(X_t-B_t)/A_t]$

$$=-\frac{\sigma^2u^2t}{2A_t^2}+t\int_{|x|\le\xi A_t}[\exp(iux/A_t)-1-iuxA_t^{-1}]\,d\nu(x)$$
$$+t\int_{|x|>\xi A_t}[\exp(iux/A_t)-1]d\nu(x)-iuM.$$

We define measures ν_t and γ_t such that for Borel $E\subset R-\{0\}$,

(2.2) $$\nu_t(E)=t\nu(A_tE)$$
$$\gamma_t(E)=\int_E x^2/(1+x^2)\,d\nu_t(x)$$

and

$$\gamma_t(\{0\})=t\sigma^2/A_t^2$$

where $A_tE=\{A_tx : x\in E\}$. Then (2. 1) can be written as

(2. 3) $$\int_{|x|\le\xi}[\exp(iux)-1-iux]\frac{1+x^2}{x^2}d\gamma_t(x)$$
$$+\int_{|x|>\xi}[\exp(iux)-1]\frac{1+x^2}{x^2}d\gamma_t(x)-iuM$$

where $[\exp(iux)-1-iux](1+x^2)/x^2$ is defined to be $-u^2/2$ at $x=0$. Similarly for $\bar{\nu}$, define for Borel $E\subset R-\{0\}$,

$$\bar{\gamma}(E)=\int_E x^2/(1+x^2)d\bar{\nu}(x)$$

and

$$\bar{\gamma}(\{0\})=\bar{\sigma}^2.$$

Then log characteristic function of Y is expressed as

(2. 4) $$ibu+\int[\exp(iux)-1-iux(1+x^2)^{-1}]\frac{1+x^2}{x^2}\,d\bar{\gamma}(x)$$

where

$$[\exp(iux)-1-iux(1+x^2)^{-1}](1+x^2)/x^2$$

is defined to be $-u^2/2$ at $x=0$. (1) and (2) can be rewritten as follows;

(1)′ $$\lim_{\varepsilon\to0}\overline{\lim_{t\to0}}\left(\int_{|x|\le\varepsilon}x^2d\nu_t(x)+t\sigma^2/A_t^2\right)$$
$$=\lim_{\varepsilon\to0}\underline{\lim_{t\to0}}\left(\int_{|x|\le\xi}x^2d\nu_t(x)+t\sigma^2/A_t^2\right)$$
$$=\bar{\sigma}^2.$$

(2)′ $$\lim_{t\to0}\int_{x>y}d\nu_t(x)=\int_{x>y}d\bar{\nu}(x)$$
$$\lim_{t\to0}\int_{x<-y}d\nu_t(x)=\int_{x<-y}d\bar{\nu}(x).$$

At conitinuity point, $-y<0$, of $\bar{\nu}$ (hence of $\bar{\gamma}$), as $t\to0$,

$$\gamma_t(-\infty,\ -y)=\int_{x<-y}x^2/(1+x^2)d\nu_t(x)$$
$$\to\int_{x<-y}x^2/(1+x^2)d\bar{\nu}(x)$$
$$=\bar{\gamma}(-\infty,-y).$$

Similarly, at continuity point, $y>0$ of $\bar{\nu}$, as $t\to0$, $\gamma_t(y,\infty)\to\bar{\gamma}(y,\infty)$. Furthermore by (1)′, we obtain

$$\lim_{\varepsilon\to0}\overline{\lim_{t\to0}}\int_{|x|\le\varepsilon}(1+x^2)d\gamma_t(x)=\lim_{\varepsilon\to0}\underline{\lim_{t\to0}}\int_{|x|\le\varepsilon}(1+x^2)d\gamma_t(x)=\bar{\sigma}^2,$$

which implies that

$$\lim_{\varepsilon\to0}\overline{\lim_{t\to0}}\ \gamma_t[-\varepsilon,\varepsilon]=\lim_{\varepsilon\to0}\underline{\lim_{t\to0}}\ \gamma_t[-\varepsilon,\varepsilon]=\bar{\sigma}^2$$

since

$$\int_{|x|\le\varepsilon}d\gamma_t(x)\le\int_{|x|\le\varepsilon}(1+x^2)d\gamma_t(x)\le(1+\varepsilon^2)\int_{|x|\le\varepsilon}d\gamma_t(x).$$

Thus at every conitinuity point y of $\bar{\gamma}$, we have as $t\to0$,

$$\gamma_t(-\infty,y)\to\bar{\gamma}(-\infty,y)$$

and

$$\gamma_t(-\infty,\infty)\to\bar{\gamma}(-\infty,\infty).$$

This means that $\gamma_t\overset{w}{\Rightarrow}\bar{\gamma}$ as $t\to0$. Thus we have proved that (2.1) converges to

$$\int_{|x|\le\xi}(\exp(iux)-1-iux)d\bar{\nu}(x)+$$
$$\int_{|x|>\xi}(\exp(iux)-1)d\bar{\nu}(x)-iuM.$$

It is clear that M is necessarily$-(\beta^{\bar{\nu}}(\xi)-\alpha^{\bar{\nu}}(\xi)+\bar{b})$ in order to obtain Y as the limit of $(X_t-B_t)/A_t$ as $t\to0$. The proof of converse is almost identical to Theorem 3.2 in [10] except replacing $-u^2/2$ by Re log $E\exp(iuY)$ and using

$$\lim_{u\to0}\text{Re}\log E\exp(iuY)=0.$$

(b) The proof is very similar to (a).

For complete characterizations of the class of limit distributions, we need two technical lemmas.

LEMMA 2 (Lemma 1 of [2]). *(a) Let U be positive and monotone on $(0,\infty)$ and suppose that there exist sequences of numbers $a_n\to0$, $\lambda_n>0$ such that*

$$\lambda_{n+1}/\lambda_n\to1,\ \textit{and}$$
$$\lambda_nU(a_nx)\to\phi(x)$$

exists on a dense set and $0<\phi(x)<\infty$ on some interval. Then $\phi(x)=Cx^\rho$ and U varies regularly at 0 where $-\infty<\rho<\infty$.

(b) The same conclusion holds if $a_n\to\infty$ where U varies regularly at ∞.

Proof. Only (b) is in [2], but (a) can be obtained from (b) if we consider $U(1/x)$.

LEMMA 3. *Let* $\sigma=0$ *and* $0<\alpha<2$.
(a) *If* G *is regulary varying at* 0 *with order* $-\alpha$, *then*

$$\lim_{x\to0}\frac{G^\nu(x)}{K^\nu(x)}=\frac{2-\alpha}{\alpha}.$$

(*b*) *If* G *is regularly varying at* ∞ *with order* $-\alpha$, *then*

$$\lim_{x\to\infty}\frac{G^\nu(x)}{K^\nu(x)}=\frac{2-\alpha}{\alpha}.$$

Proof. (a) Denote G^ν and K^ν by G and K. Integrating by parts, we have

$$\int_{z<|y|\le x}y^2d\nu(y)=-x^2G(x)+z^2G(z)+2\int_z^x yG(y)\,dy.$$

Since for x sufficiently small, $G(x)=x^{-\alpha}l(x)$ where l is slowly varying at zero,

$$\int_{|y|\le x}y^2d\nu(y)=-x^2G(x)+2\int_0^x yG(y)\,dy.$$

Now using the fact that $G(1/x)$ is regularly varying at ∞ with order α and Therorem 1 in [1], VIII. 9, we obtain for x sufficiently small,

$$\int_0^x yG(y)\ dy\sim\frac{1}{2-\alpha}x^2G(x).$$

(b) Integrating by parts,

$$x^2K(x)-K(1)=-x^2G(x)+G(1)+2\int_1^x yG(y)\,dy.$$

It is obvious that $\int x^2d\nu(x)=\infty$ by the regular variation of G at ∞. Again using Theorem 1 in [1], VIII. 9,

$$\int_1^x yG(y)\ dy\sim\frac{1}{2-\alpha}x^2G(x).$$

Now we are ready to prove our main theorem.

THERốREM 1. (*a*) (1) *The class of limit distributions of* $\{(X_t-B_t)/A_t\}$ *as* $t\to0$ *for suitable numbers* $\{A_t\}$, $\{B_t\}$ *coincides with the family of stable distributions.*

(2) *For some* $\{A_t\}$, $\{B_t\}$, *the limit of* $(X_t-B_t)/A_t$ *as* $t\to0$ *has stable distribution with index* α, $0<\alpha<2$ *iff* G^ν *is regularly varying at zero*

with order $-\alpha$ *and* $G_+^{\nu}(x)/G^{\nu}(x)\to C_1/(C_1+C_2)$, $G_-^{\nu}(x)/G^{\nu}(x)\to C_2/(C_1+C_2)$ *as* $x\to 0$, *for some* $C_1\ C_2\geqslant 0$.

(*b*) *The same conclusions hold if* $t\to 0$ *and* $x\to 0$ *are replaced by* $t\to\infty$ *and* $x\to\infty$ *respectively.*

Proof. (a) By the central convergence criterion it is obvious that there exist a Lèvy process $\{X_t\}$, and numbers $\{A_t\}$, $\{B_t\}$ such that $(X_t-B_t)/A_t$ converges to a stable distribution. It remains to show that the limit has stable distribution if there exist such $\{A_t\}$ and $\{B_t\}$. As we did before, denote the Lèvy represntation of characterstic function of Y by $(\bar{b},\bar{\sigma},\bar{\nu})$. Since $tG_+^{\nu}(A_tx)\to G_+^{\bar{\nu}}(x)$ and similarly for $G^{\bar{\nu}}_-$ and G^{ν} at continuity points of $\bar{\nu}$, by Lemma 2, either $G^{\nu}(x)=0$ for any $x>0$ or $G_+{}^{\nu}(x)=C_1x^{-\alpha}$ and $G_-^{\bar{\nu}}(x)=C_2x^{-\alpha}$ for any $x>0$ must hold where $C_1+C_2>0, C_1\geqslant 0,\ C_2\geqslant 0$. The first possiblity leads to the normal or degenerate distribution. If the second possibility occurs, then $0<\alpha<2$ holds since $\int x^2/(1+x^2)d\nu(x)<\infty$. And by Lemma 2, G_+^{ν} and G_-^{ν} are regularly varying at 0 with order $-\alpha$. If we write $G^{\nu}(x)$ as $x^{-\alpha}l(x)$ where l is slowly varying at 0, then by (2) in Lemma 1, at continuity point y of $\bar{\nu}$,

$$\lim_{t\to 0}\frac{tl(yA_t)}{(yA_t)^{\alpha}}=(C_1+C_2)/y^{\alpha}$$

which implies that

$$\lim_{t\to 0} t/A_t^2=\infty. \tag{2.5}$$

By Lemma 3, for t sufficiently small

$$\varepsilon^2tK^{\nu}(\varepsilon A_t)\sim\frac{t\sigma^2}{A_t^2}+\varepsilon^2t\frac{\alpha}{2-\alpha}G^{\nu}(\varepsilon A_t) \tag{2.6}$$

and by (2) in Lemma 1,

$$\lim_{\varepsilon\to 0}\varliminf_{t\to 0}\varepsilon^2tG^{\nu}(\varepsilon A_t)=\lim_{\varepsilon\to 0}\varlimsup_{t\to 0}\varepsilon^2tG^{\nu}(\varepsilon A_t)=0. \tag{2.7}$$

Thus (1) in Lemma 1, (2.5) and (2.7) imply that as $t\to 0$,

$$\sigma^2t/A_t^2\to\bar{\sigma}^2,$$

but (2.5) forces $\sigma^2=\bar{\sigma}^2=0$. Since $G_+^{\bar{\nu}}(x)=C_1x^{-\alpha}$ and $G_-^{\bar{\nu}}(x)=C_2x^{-\alpha}$, it is clear that $G_+^{\nu}(x)/G^{\nu}(x)\to C_1/(C_1+C_2)$, $G_-^{\nu}(x)/G^{\nu}(x)\to C_2/(C_1+C_2)$ as $x\to 0$. The converse is obvious from the context of proof given above. (b) The proof is again similar to (a) but here $t/A_t^2\to 0$ as $t\to\infty$, hence $\sigma^2=0$ is not necessarily true even though $\bar{\sigma}^2=0$ is still true for $0<\alpha<2$.

3. Stochastic Compactness

Recall the definition of g and a_t under the assumption $\int d\nu(x) = \infty$ from Section 1. Since there are no other Lèvy measure than ν arising in this section, we write G and K for G^ν and K^ν respectively and do similarly for other functions. We start with some observations about g.

LEMMA 4. (Lemma 2.4 in [9]) *If $G(x) \leq CK(x)$ on intrval I for some constant C, then $x^\delta g(x)$ is decreasing on I where $\delta = 2/(C+1) < 2$.*

Proof. It suffices to prove the statement when $\sigma^2 = 0$ and then the proof is identical to Lemma 2.4 in [9].

Lemma 4 and the fact that $x^2 g(x) \uparrow$ give control over the growth of $g(x)$. If $G(x) \leq CK(x)$ on interval I, then on I for $0 < \eta < 1$,

$$\eta^{-\delta} g(x) \leq g(\eta x) \leq \eta^{-2} g(x) \tag{3.1}$$

and for $M > 1$

$$M^{-2} g(x) \leq g(Mx) \leq M^{-\delta} g(x). \tag{3.2}$$

Now we obtain three equivalent statements concerning stochastic compactness in appropriate sense.

THEOREM 2. (*a*) *Following three statements are equivalent;*
(1) *There exist $\{A_t\}$, $\{B_t\}$ such that for any $t_n \to 0$, $\{(X_{t_n} - B_{t_n})/A_{t_n}\}$ has further subsequence which converges to a nondegenerate distribution.*

(2) (i) $\lim_{T\to\infty} \overline{\lim_{t\to 0}}\, tG(Ta_t) = 0.$
(ii) $\underline{\lim_{t\to 0}}\, tK(\xi a_t) > 0$
for some $\xi > 0$.

(3) $$\overline{\lim_{x\to 0}} \frac{G(x)}{K(x)} < \infty.$$

(*b*) *The statements similar to* (1), (2) *and* (3) *for* $t \to \infty$ *and* $x \to \infty$ *and equivalent.*

Proof. (a) (1) implies (2): It is well-known that $\{(X_{t_n} - B_{t_n})/A_{t_n}\}$ for $t_n \to 0$ has further subsequence which converges to a distribution (not necessarily nondegenerate) iff

$$\lim_{T\to\infty} \overline{\lim_{t\to 0}}\, P(|X_t - B_t| > TA_t) = 0. \tag{3.3}$$

Consider a symmetric process $X_t^s = X_t - \tilde{X}_t$ where

$\mathcal{F}\{X_t, t \geqslant 0\}$ and $\mathcal{F}\{\tilde{X}_t, t \geqslant 0\}$ are independent and $\{X_t\}$ has same distribution as $\{\tilde{X}_t\}$. It is clear that if (3.3) holds, then

(3.4) $$\lim_{T\to\infty} \overline{\lim_{t\to 0}}\, P(|X_t^s|>TA_t)=0.$$

Then by Lèvy's Inequality, we have

$$\begin{aligned} P(|X_t^s|>TA_t) &\geqslant 2^{-1}P\ (\sup_{u\le t}|X_u^s|>TA_t) \\ &\geqslant 2^{-1}[1-\exp(-tG^s(2TA_t)] \\ &=2^{-1}[1-\exp(-2tG(2TA_t)] \end{aligned}$$

since

$P(\sup_{u\le t} |X_u^s|\le TA_t)\le P$(there exist no jumps of magnitude greater than $2TA_t$ up to time t)

where G^s denote the corresponding G function for $\{X_t^s\}$. By (3.4), it is clear that

(3.5) $$\lim_{T\to\infty} \overline{\lim_{t\to 0}}\, tG(TA_t)=0.$$

Also one may easily see that for some $\xi>0$,

(3.6) $$\underline{\lim_{t\to 0}}\, tK(\xi A_t)>0.$$

since otherwise there exists a sequence converging to a degenerate distribution under (i) of (2). We observe that there exists C such that for $0<C<C'$

(3.7) $$\overline{\lim_{t\to 0}}\, tg(C'A_t)\le\lim_{t\to 0} t\, g(CA_t)<\infty$$

by the central convergence criterion, Lemma 1. (3.6) and (3.7) imply that there exist $0<m<M<\infty$ such that for t sufficiently small,

$$m/t\le g(\xi\ A_t)\le M/t$$

hence

$$a_{t/m}\ge\xi A_t\ge a_{t/M}.$$

By the comparability of A_t and a_t, and (3.5) and (3.6), (2) clearly holds.

(2) implies (3): Suppose that $\overline{\lim_{x\to 0}}\, \frac{G(x)}{K(x)}=\infty$ i.e. $\underline{\lim_{x\to 0}}\, \frac{K(x)}{G(x)}=0.$

We consider two possibilities;

$$\overline{\lim_{x\to 0}}\, \frac{K(x)}{G(x)}=0 \quad\text{or}\quad \overline{\lim_{x\to 0}}\, \frac{K(x)}{G(x)}>0.$$

But the first possibility is easily ruled out, since then

$$\lim_{t\to 0} \frac{tK(\xi a_t)}{tG(\xi a_t)}=0$$

which is contradiction to (2). Assume that

$$\overline{\lim_{x\to 0}} \frac{K(x)}{G(x)}>0.$$

Choose $x_i \downarrow 0$ so that for $\varepsilon_i \downarrow 0$,

$$\frac{K(x_i)}{G(x_i)}\leq \varepsilon_i, \text{ and } g(x_i)\to\infty.$$

Let $yi=\sup \{y<x_i; K(y)/G(y)>\varepsilon_i\}$ and $g(x_i)=1/t_i$, $g(y_i)=1/s_i$. Then $x_i=a_{t_i}$ $y_i=a_{s_i}$ and there exist $u_i\to 0$ such that $a_{s_i}\leq \xi a_{u_i}\leq a_{t_i}$. By definition of y_i, $K(\xi\ a_{u_i})/G(\xi\ a_{u_i})\leq\varepsilon_i$ which is again contradiction to (2).

(3) implies (1): Recall that there exists $0<\delta<2$ such that $x^\delta\downarrow g(x)$ for x small enough under the assumption (3). Let $\lambda^\delta>4$, $T>\lambda$ and $b_t=t[b+\beta(\lambda a_t)-\alpha(\lambda a_t)]$. Write $X_t$as a sum of independent processes, $X_t{}^1$, $X_t{}^2$ and $t(b-\alpha(Ta_t))$ where

$$\log E \exp\ (iuX_t{}^1)=t\Big[-\frac{\sigma^2u^2}{2}+\int_{|x|\leq Ta_t}(\exp(iux)$$
$$-1-iux(1+x^2)^{-1})d\nu(x)\Big]$$
$$\log E\exp\ (iu\ X_t^2)=t\Big[\int_{|x|>Ta_t}(\exp(iux)-1)d\nu(x)\Big].$$

Then by differentiating the characteristic function, it is obvious that $EX_t{}^1=t\beta(Ta_t)$ and $\mathrm{Var}X_t{}^1=tT^2a_t{}^2K(Ta_t)$. Note that for t small enough,

$$(3.8)\qquad t|\beta(Ta_t)-\beta(\lambda a_t)|\leq t\int_{\lambda a_t<|x|\leq Ta_t}|x^3/(1+x^2)|d\nu(x)$$
$$<tT^3a_t{}^3G(\lambda a_t)$$
$$<\lambda^{-\delta}T^3a_t{}^3$$
$$<Ta_t/4,$$

and

$$(3.9)\qquad t|\alpha(Ta_t)-\alpha(\lambda\ a_t)|\leq t\int_{\lambda a_t<|x|\leq Ta_t}|x/(1+x^2)|d\nu(x)$$
$$<tTa_tG(\lambda a_t)$$
$$<\lambda^{-\delta}Ta_t$$
$$<Ta_t/4.$$

Using (3.8) and (3.9), and Chebychev's Inequality,

$$\begin{aligned} P(|X_t-b_t|\geqslant Ta_t) &\leqslant P(|X_t^1+t[\alpha\ (\lambda a_t)-\alpha(Ta_t)-\beta(\lambda a_t)]|\geqslant Ta_t) \\ &\quad +1\text{-}\exp(-tG(Ta_t)) \\ &\leq P(|X_t^1-t\beta(Ta_t)|\geqslant Ta_t/2)+tG(Ta_t) \\ &\leq 4tK(Ta_t)+tG(Ta_t) \\ &< 4T^{-\delta}, \end{aligned}$$

for t sufficiently small. It remains to show that every subsequential limit of $(X_t-b_t)/a_t$ has nondegenerate distribution. But this is easy to see by the central convergence criterion since for t small enough and some constant $C>0$, $\xi>1$,

$$tK(\xi a_t)\geqslant Ctg(\xi a_t)\geqslant C\xi^{-2}>0.$$

(b) It is similar to (a).

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