J. Korean Math. Soc. 22 (1985), No. 2, pp. 191-197

CONVOLUTIONS OF MEASURES ATTRACTED TO STABLE MEASURES ON BANACH SPACES

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1. Introduction and Notation

Let B and B^* denote a real separable Banach space with the norm $\|\cdot\|$ and its topological dual, respectively. By $\langle x, x'\rangle$ we shall denote the dual pairing between B and B^*. By a probability (prob.) measure on B we shall always mean that it is defined on $\mathscr{B}(B)$, i.e. the smallest σ-algebra containing all the open sets (in norm topology) of B. If μ and ν are two prob. measures on B, the *convolution* of μ and ν is defined by

$$\mu * \nu(E) = \int_B \mu(E-x)\,d\nu(x)$$

for every $E \in \mathscr{B}(B)$. The symbol μ^{*n} will denote μ convoluted n times with itself. For $x \in B$, $\delta(x)$ will denote degenerate prob. measure concentrated at the point x. By $\mathbf{R}$ and $\mathbf{R}^+$ we shall denote the set of real numbers and positive real numbers, respectively. For any $a \in \mathbf{R}^+$, $T_a\mu$ is defined to be the prob. measure on B given by $T_a\mu(E) = \mu(a^{-1}E)$ for every $E \in \mathscr{B}(B)$. The *characteristic functional* of μ is defined by the formula

$$\hat{\mu}(x') = \int_B e^{i\langle x, x'\rangle}\,d\mu(x)$$

for every $x' \in B^*$. It is well known that the *characteristic functional* determines uniquely the prob. measure.

We say that μ is *stable (prob.) measure of index* α $(0 < \alpha \leq 2)$ on B if for any $a, b, \in \mathbf{R}^+$, there exist a number $c \in \mathbf{R}^+$ with $c^\alpha = (a^\alpha + b^\alpha)$ and an element $x \in B$ such that

$$T_a\mu * T_b\mu = T_c\mu * \delta(x).$$

A sequence $\{\mu_n\}$ of prob. measures on B is said to converge weakly a

This research was supported by a grant from MOE, 1984.
Received June 1, 1985.

prob. measure μ (in symbols, $\lim_{n\to\infty} \mu_n=\mu$ or $\mu_n\to\mu$) on B if for every bounded continuous real valued function f on B

$$\int_B f\, d\mu_n \to \int_B f\, d\mu.$$

We say that a non-degenerate probability measure ν on B belongs to the domain of attraction of a non-degenerate prob. measure μ of index $\alpha(0<\alpha\leq 2)$, denoted by $\nu\in D(\alpha)$ if there exist a sequence $\{a_n\}$ in $\mathbf{R}^+$, called normalizing coefficients, and a sequence $\{x_n\}$ in B, called centering vectors such that

$$\lim_{n\to\infty} T_{a_n}\nu^{*n} * \delta(x_n)=\mu.$$

It is known in [3] that stable measures and only stable measures have nonempty domain of attraction.

The purpose of this paper is to show that if $\lambda\in D(\alpha)$ and $\nu\in D(\beta)$ where $0<\alpha\leq\beta\leq 2$, and if $\{a_n\}$ and $\{b_n\}$ are normalizing coefficients respectively of λ and ν, then $\lambda * \nu\in D(\alpha)$ and its normalizing coefficients are $\{(a_n{}^{-\alpha}+b_n{}^{-\alpha})^{-1/\alpha}\}$. This result extends a result, due to Tucker [7] proved in real line $\mathbf{R}$ to a real separable Banach space setting.

2. Preliminaries

In this section we collect necessary definitions and some known results which will be needed in this paper.

DEFINITION 2.1. A function U is said to be *regulary varying with exponent* γ $(-\infty<\gamma<\infty)$ if it is real-valued, positive and measurable on $[a,\infty)$ for some $a>0$, and if for each $x>0$,

$$\lim_{t\to\infty}\frac{U(tx)}{U(t)}=x^{\gamma}.$$

A function L which is regulary varying with $\gamma=0$ is called *slowly varying.*

It can be shown that $U(\cdot)$ is regulary varying if and only if it can be written in the form

$$U(x)=x^{\gamma}L(x)$$

where $-\infty<\gamma<\infty$ and $L(\cdot)$ is slowly varying.

LEMMA 2.2 [5]. *Let L, L_1 and L_2 denote slowly varying functions. Then the following holds*

(1) *for any* $\delta>0$, $x^{-\delta}L(x)\to 0$ *as* $x\to\infty$.

(2) L^{δ} *for any* $\delta\in\mathbf{R}$, L_1+L_2, $L_1\cdot L_2$ *and* L_1/L_2 *are also slowly varying functions.*

LEMMA 2.3 [2]. *Let* μ *be a prob. measure on* B. *Then* μ *is stable of index* α $(0<\alpha\leq 2)$ *if and only if either, for every* $x'\in B^*$

(1) $\hat{\mu}(x')=\exp\{i\langle x_0, x'\rangle-\frac{1}{2}\langle Tx', x'\rangle\}$ *if* $\alpha=2$

where $x_0\in B$ *and* T *is the covariance operator from* X^* *into* X, *or*

(2) *there exists a finite Borel measure* Γ *on* $S=\{x\in B : \|x\|=1\}$ *and a vector* $x_0\in B$ *such that for every* $x'\in B^*$

$$\hat{\mu}(x')=\exp\{i\langle x_0, x'\rangle-\int_S\langle s, x'\rangle d\Gamma(s)+iC(\alpha, x')\}$$

where

$$C(\alpha, x')=\begin{cases}\tan\dfrac{\pi\alpha}{2}\displaystyle\int_S\langle s, x'\rangle\,|\langle s, x'\rangle|^{\alpha-1}\,d\Gamma(s) & \text{if } \alpha\neq 1\\ \dfrac{2}{\pi}\displaystyle\int_S\langle s, x'\rangle\log|\langle s, x'\rangle|\,d\Gamma(s) & \text{if } \alpha=1.\end{cases}$$

If μ is a stable measure on B of index α, in view of Lemma 2.3, we shall denote it by $\mu=[2, x_0, T]$ when $\alpha=2$, and by $\mu=[\alpha, x_0, \Gamma]$ when $0<\alpha<2$.

Let μ be an infinitely divisible prob. measure on B. Then it is known [6] that for any $t>0$, there exists an infinitely divisible prob. measure ν, denoted by μ^t, on B such that $\hat{\nu}(x')=(\hat{\mu}(x'))^t$ for every $x'\in B^*$.

LEMMA 2.4 [1]. *Let* μ *be an infinitely divisible prob. measure on* B. *Then* μ *is stable of index* $\alpha(0<\alpha\leq 2)$ *if and only if for any* $t>0$, *there exists a vector* $x_t\in B$ *such that* $\mu^t=T_{t^{-1/\alpha}}\mu*\delta(x_t)$.

3. Convolutions of measures in the domain of attraction of stable measures and their normalizing coefficients.

In this section we investigate the following problem: If $\lambda\in D(\alpha)$ and $\nu\in D(\beta)$ where $0<\alpha\leq\beta\leq 2$, and if $\{a_n\}$ and $\{b_n\}$ are normalizing coefficients, then what can be said about the domain of attraction and the normalizing coefficients for $\lambda*\nu$? We first begin with a lemma.

LEMMA 3.1. *Let* ν *be in* $D(\alpha)$ *with normalizing coefficients* $\{a_n\}$ *and* *centering vectors* $\{x_n\}$. *Then there exists a slowly varying function* L *defined on* $[1, \infty)$ *such that* $a_n = n^{-1/\alpha} L(n)$ *for every positive integer* n.

Proof. For any $u \geq 1$, we define $b(u) = a_{[u]}$, where $[u]$ denotes the integral part of u. Then for any $t > 0$ and $x' \in B^*$, we have

$$\hat{\nu}^{[ut]}(b(ut)x') \cdot e^{i\langle x_{[ut]}, x'\rangle}$$

$$= \left(\hat{\nu}^{[u]}\left(b(u) \cdot \frac{b(ut)}{b(u)} x'\right) \cdot e^{i\left\langle \frac{b(ut)}{b(u)} x_{[u]}, x'\right\rangle}\right)^{[ut]/[u]} \cdot e^{i\langle z_{[u]}, x'\rangle}$$

where $z_{[u]} = x_{[ut]} - \dfrac{[ut]}{[u]} \cdot \dfrac{b(ut)}{b(u)} x_{[u]}$. Since

$$\lim_{u \to \infty} \hat{\nu}^{[ut]}(b(ut)x') \cdot e^{i\langle x_{[ut]}, x'\rangle} = \hat{\mu}(x')$$

for every $x' \in B^*$ and $[ut]/[u] \to t$, it follows that for every $x' \in B^*$,

$$\lim_{u \to \infty} (\hat{\nu}^{[u]}(b(u)x') \cdot e^{i\langle x[u], x'\rangle})^{[ut]/[u]} = \hat{\mu}^t(x').$$

Hence by using the same arguments as in [1, p. 214], there exist $c(t) \in \mathbf{R}^+$ and $x_t \in B$ such that

$$\frac{b(ut)}{b(u)} \to c(t) \text{ and } \mu = T_{c(t)}\mu^t * \delta(x_t).$$

From this along with Lemma 2.4 we have $c(t) = t^{-1/\alpha}(0 < \alpha \leq 2)$. This shows that $b(\cdot)$ is a regularly varying function on $[1, \infty)$. Hence there exists a slowly varying function L on $[1, \infty)$ such that $b(u) = u^{-1/\alpha} L(u)$. Therefore we have a slowly varying function L on $[1, \infty)$ such that $a_n = n^{-1/\alpha} L(n)$ for every positive integers n.

THEOREM 3.2. *Let* $\lambda \in D(\alpha)$ *and* $\nu \in D(\beta)$, *where* $0 < \alpha < \beta \leq 2$, *and let* $\{a_n\}$ *be normalizing coefficients for* λ. *Then* $\lambda * \nu \in D(\alpha)$ *and* $\{a_n\}$ *are normalizing for* $\lambda * \nu$.

Proof. Since $\lambda \in D(\alpha)$ and $\nu \in D(\beta)$, there exist sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbf{R}^+$, and sequences $\{x_n\}$ and $\{y_n\}$ in B for λ and ν, respectively such that

$$\lim_{n \to \infty} T_{a_n}\lambda^{*n} * \delta(x_n), \text{ and } \lim_{n \to \infty} T_{b_n}\nu^{*n} * \delta(y_n)$$

are stable measures of indices α and β, respectively. Now let $\mu = \lambda * \nu$ and $z_n = x_n + (a_n/b_n)y_n$. Then, for each n

$$\begin{aligned} T_{a_n}\mu^{*n} * \delta(z_n) &= T_{a_n}\lambda^{*n} * T_{a_n}\nu^{*n} * \delta(x_n) * \delta((a_n/b_n)y_n) \\ &\doteq T_{a_n}\lambda^{*n} * \delta(x_n) * T_{a_n/b_n}(T_{b_n}\nu^{*n} * \delta(y_n)). \quad (3.3) \end{aligned}$$

From Lemma 3. 1 there exist slowly varying functions L_1 and L_2 on $[1, \infty)$ such that $a_n=n^{-1/\alpha}L_1(n)$ and $b_n=n^{-1/\beta}L_2(n)$ for every positive integers n. Hence we have

$$\frac{a_n}{b_n}=n^{(1/\beta)-(1/\alpha)}\cdot\frac{L_1(n)}{L_2(n)}.$$

Since $(1/\beta)-(1/\alpha)<0$, it follows from Lemma 2. 2 that we have $a_n/b_n \to 0$ as $n\to\infty$. From the equation (3. 3) and the continuity of convolution ([4], p. 57), it follows that

$$\lim_{n\to\infty} T_{a_n}\mu^{*n} * \delta(z_n)=\lim_{n\to\infty} T_{a_n}\lambda^{*n} * \delta(x_n)$$

This shows that $\mu=\lambda * \nu \in D(\alpha)$ with normalizing coefficients $\{a_n\}$.

THEOREM 3. 3. *If $\lambda\in D(\alpha)$ and $\nu\in D(\beta)$, where $0<\alpha\leq\beta\leq 2$, and if $\{a_n\}$ and $\{b_n\}$ are normalizing coefficients for λ and ν, respectively, then $\lambda * \nu\in D(\alpha)$ and $\{(a_n^{-\alpha}+b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients for $\lambda * \nu$.*

Proof. Case(i) $0<\alpha<\beta\leq 2$. Since $a_n/b_n\to 0$ as $n\to\infty$ in the proof of Theorem 3. 3 we have

$$a_n^{-\alpha} \sim a_n^{-\alpha}+b_n^{-\alpha}$$

where the notation $u_n \sim v_n$ means that $u_n/v_n\to 1$ as $n\to\infty$. Hence from Theorem 3. 3 it follows that $\lambda * \nu\in D(\alpha)$ and $\{(a_n^{-\alpha}+b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients for $\lambda * \nu$. Case(ii) $0<\alpha=\beta\leq 2$. Since $\lambda, \nu\in D(\alpha)$, we can choose the centering vectors $\{x_n\}$ and $\{y_n\}$ in B such that

$$\lim_{n\to\infty} T_{a_n}\lambda^{*n} * \delta(x_n)=\eta_1 \text{ and } \lim_{n\to\infty} T_{b_n}\nu^{*n} * \delta(y_n)=\eta_2$$

are stable measures of index α with $x_0=0$ in the representation of Lemma 2. 3. First, let η_1 and η_2 be stable measures of index 2 with the representations $[2, 0, T_1]$ and $[2, 0, T_2]$, respectively. Then for any $a, b\in\mathbf{R}^+$ $T=a^2T_1+b^2T_2$ is a covariance operator from B^* into B. Hence there exists a stable measure η of index 2 with the representation $\eta=[2, 0, T]$. Next, let η_1 and η_2 be stable measures of index $\alpha(0<\alpha<2)$ with the representations $[\alpha, 0, \Gamma_1]$ and $[\alpha, 0, \Gamma_2]$, respectively. Then for any $a, b\in\mathbf{R}^+$ $\Gamma=a^\alpha\Gamma_1+b^\alpha\Gamma_2$ is a finite measure on S. Hence it follows from Lemma 2. 3 that there exists a stable measure η of index α with the representation $\eta=[\alpha, 0, a^\alpha\Gamma_1+b^\alpha\Gamma_2]$. Now for any $a, b\in\mathbf{R}^+$ let $\Gamma_3=\frac{2}{\pi}((a\log a)\Gamma_1+(b\log b)\Gamma_2)$. Since Γ_1 and Γ_2 are finite measure on S and so

$$\int_S \|s\| d\Gamma_i(s) < \infty, \quad i=1,2$$

Thus for each $i=1,2$, there exists $y_i \in B$ such that

$$y_i = B\int_S s\, d\Gamma_i(s)$$

where B stands for the Bochner integral with respect to Γ_i on S. Hence there exists $y_0 \in B$ such that for every $x' \in B^*$,

$$\int_S \langle s, x' \rangle d\Gamma_3(s) = \langle y_0, x' \rangle.$$

In either case $\alpha=2$ or $0<\alpha<2$, for any $a, b \in \mathbf{R}^+$ there exists a stable measure η of index α such that for any $x' \in B^*$

$$\widehat{(T_a\eta_1 + T_b\eta_2)}(x') = \hat{\eta}_1(ax')\hat{\eta}_2(bx') = \begin{cases} \hat{\eta}(x') & \text{if } \alpha \neq 1 \\ \hat{\eta}(x') \exp\{i\langle y_0, x' \rangle\} & \text{if } \alpha = 1. \end{cases}$$

Now let L_1 and L_2 be slowly varying functions on $[1, \infty)$ such that $a_n = n^{-1/\alpha} L_1(n)$ and $b_n = n^{-1/\alpha} L_2(n)$ for every positive integers n. Let's define a function L on $[1, \infty)$ by

$$L(x) = (L_1^{-\alpha}(x) + L_2^{-\alpha}(x))^{-1/\alpha}.$$

Then by Lemma 2.2, L is a slowly varying function on $[1, \infty)$. Now let $\mu = \lambda * \nu$, and for each n let $p_n = L(n)/L_1(n)$, $q_n = L(n)/L_2(n)$, $c_n = n^{-1/\alpha} L(n)$, and

$$\mu_n = T_{c_n}\mu^{*n} * \delta(p_n x_n + q_n y_n - h(\alpha))$$

where $h(\alpha) = y_0$ if $\alpha = 1$, and $h(\alpha) = 0$ if $\alpha \neq 1$. Then we have, for each $n = 1, 2, \ldots$, $p_n^\alpha + q_n^\alpha = 1$ and

$$\mu_n = T_{p_n}(T_{a_n}\lambda^{*n} * \delta(x_n) * T_{q_n}(T_{b_n}\nu^{*n} * \delta(y_n) * \delta(-h(\alpha)). \quad (3.4)$$

Let's write

$$\lambda_n = T_{a_n}\lambda^{*n} * \delta(x_n), \quad \nu_n = T_{b_n}\nu^{*n} * \delta(y_n).$$

Then we have $\mu_n = T_{p_n}\lambda_n * T_{q_n}\nu_n * \delta(-h(\alpha))$ for each $n=1, 2, \ldots$

We now shall show that $\mu_n \to \eta$ as $n \to \infty$. To show this, we first show that $\{\mu_n\}$ is conditionally compact. Let $\{\mu_m\}$ be a subsequence of $\{\mu_n\}$. Since $\{p_n\}$ is a bounded sequence in $\mathbf{R}^+$, the corresponding subsequence $\{p_m\}$ of $\{p_n\}$ to $\{\mu_m\}$ has a convergent subsequence $\{p_{m(k)}\}$ of $\{p_m\}$ and so the corresponding subsequence $\{q_{m(k)}\}$ of $\{q_m\}$ is also convergent. Thus there exists $a \geq 0$ and $b \geq 0$ such that $p_{m(k)} \to a$ and

$q_{m(k)} \to b$ as $k \to \infty$. Hence from (3.4), we have

$$\lim_{k\to\infty} \mu_{m(k)} = T_a \eta_1 * T_b \eta_2 * \delta(-h(\alpha)) = \eta.$$

This shows that $\{\mu_n\}$ is conditionally compact. We next show that $\hat{\mu}_n(\cdot)$ converges to $\hat{\eta}(\cdot)$. Since

$$\widehat{T_{p_{m(k)}} \lambda_{m(k)}}(x') \to \hat{\eta}_1(ax'), \quad \widehat{T_{q_{m(k)}} \lambda_{m(k)}}(x') \to \hat{\eta}_2(bx')$$

for every $x' \in B^*$, we have

$$\hat{\mu}_{m(k)}(x') \to \hat{\eta}_1(ax') \hat{\eta}_2(bx') \exp\{-h(\alpha)\} = \hat{\eta}(x').$$

Thus the subsequence $\{\hat{\mu}_m(\cdot)\}$ of $\{\hat{\mu}_n(\cdot)\}$ has further convergent subsequence $\{\hat{\mu}_{m(k)}(\cdot)\}$ which converges to $\hat{\eta}(\cdot)$. Hence $\hat{\mu}_n(\cdot) \to \hat{\eta}(\cdot)$ as $n \to \infty$. Consequently, by Lemma 2.1 ([4], p. 153), $\{\mu_n\}$ converges weakly to a stable measure η of index α. This shows that $\lambda * \nu \in D(\alpha)$ and $\{(a_n^{-\alpha} + b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients of $\lambda * \nu$.

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