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AN EXTREMAL PROBLEM FOR HOLOMORPHIC FUNCTIONS AND ITS APPLICATION

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1. Introduction

M. S. Robertson [10] established the following solution of an extremal problem. Let $\mathcal{P}$ denote the class of regular function $p(z)$, $p(0)=1$ and $\operatorname{Re} p(z)>0$, in $|z|<1$, where

$$p(z)=1+p_1z+p_2z^2+\cdots+p_nz^n+\cdots$$

Let $F(w_1, w_2)$ be a given function analytic in the half-plane $\operatorname{Re} w_1>0$ and in the w_2-plane. Then for given every r, $0<r<1$, the value of

$$\min_{p\in\mathcal{P}} \min_{|z|=r} \operatorname{Re} F(p(z), zp'(z))$$

occurs only for an extremal function of the form

$$p(z)=\sum_{n=1}^{2} \lambda_n\left(\frac{1+\varepsilon_n z}{1-\varepsilon_n z}\right)$$

where $$|\varepsilon_n|=1,\ 0\leqq\lambda_n\leqq 1,\ \sum_{n=1}^{2}\lambda_n=1.$$

Robertson's method used variational formulas depends upon the works of Hummel [4], and Schiffer [14]. The same method yields the solution of a related problem in which the term $zp'(z)$ is replaced by

$$\mu(z)=\frac{1}{z}\int_0^z p(t)dt \quad (p(z)\in\mathcal{P}). \tag{1.1}$$

It should be noticed that $\mu(z)\in\mathcal{P}$, since

$$\operatorname{Re}\mu(z)=\int_0^1 \operatorname{Re} p(\rho z)d\rho>0 \quad (z=re^{i\theta},\ r<1).$$

Specifically, we consider the following problem: Let $F(w)$ denote an arbitrary function regular in the portion D_0 of the half-plane $\operatorname{Re} w>0$

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that is covered by the images of $|z|<1$ by the mappings $w=\mu(z)$ for $p\in\mathcal{P}$. As we proceed, D_0 will be explicitly determined. We set

$$m_F(r)=\min_{p=\mathcal{P}}\min_{|z|=r}\text{Re } F(\mu(z)) \tag{1.2}$$

and consider the problem of finding its values by determining the extremal function $p(z)$. Robertson solved this problem first by the variational method. The method used by him is quite long and somewhat difficult. Its details suggested a much shorter, clearer proof by the method of subordination.

It is the purpose of this paper to present this shorter derivation of the solution of (1.2) (see Theorem 1) and to make applications (see Corollaries 1 and 2) to the class $\mathcal{R}_\alpha$ of analytic functions

$$f(z)=e^{i\alpha}z+a_2z^2+\cdots+a_nz^n+\cdots \quad (\alpha \text{ real and fixed}) \tag{1.3}$$

that are regular and satisfy the condition Re $f'(z)>0$ in $|z|<1$. It is well known [8] that such functions $f(z)$ are univalent in $|z|<1$. MacGregor [6] discussed the class $\mathcal{R}_\alpha$ in the special case $\alpha=0$. In particular, he pointed out that for $|z|<1$

$$\frac{1-|z|}{1+|z|}\leqq\text{Re } f'(z)\leqq|f'(z)|\leqq\frac{1+|z|}{1-|z|}, \tag{1.4}$$

$$-|z|+2\log(1+|z|)\leqq|f(z)|\leqq-|z|-2\log(1-|z|), \tag{1.5}$$

It is desirable to discuss the properties of a special univalent, starlike function $S(z)$ that will be useful in the proof of our theorems. The starlike character of $S(z)$ is fundamental to the proof of our Theorem 1 when we use the method of subordination.

LEMMA 1. *Let $K(z)$ denote the Koebe function $z(1-z)^{-2}$, which is univalent and starlike with respect to the origin for $|z|<1$. Then the function*

$$S(z)=\frac{2}{z}\int_0^z K(t)\,dt=\frac{2}{1-z}+\frac{2}{z}\log(1-z)=\sum_{n=1}^{\infty}\frac{2n}{n+1}z^n,$$

is also univalent and starlike with respect to the origin for $|z|<1$.

Proof. Since

$$S(z)=\frac{2}{1-z}+\frac{2}{z}\log(1-z)=\sum_{n=1}^{\infty}\frac{2n}{n+1}z^n,$$

$S(z)$ may be written in the useful alternate form

$$S(z)=\frac{1+z}{1-z}-\frac{1}{z}\int_0^z\frac{1+t}{1-t}\,dt=P_0(z)-\mu_0(z), \tag{1.6}$$

where both $P_0(z)=\dfrac{1+z}{1-z}$ and $\mu_0(z)=\dfrac{1}{z}\displaystyle\int_0^z P_0(t)\,dt$

are members of the class $\mathcal{P}$. A computation of

$$S(e^{i\theta})=U(\theta)+iV(\theta)$$

for $0<\theta\leqq\pi$ gives

$$U(\theta)=1+(\cos\theta)\log\{2(1-\cos\theta)\}-(\pi-\theta)\sin\theta, \quad (1.7)$$

$$\begin{aligned} V(\theta)&=(\sin\theta)(1-\cos\theta)^{-1}-(\pi-\theta)\cos\theta-\sin\theta\log\{2(1-\cos\theta)\}\\ &=\frac{dU(\theta)}{d\theta}. \end{aligned} \quad (1.8)$$

For $0<\theta<\pi$, we obtain from (1.6) the inequalities

$$U(\theta)=-\mathrm{Re}\left[\frac{1}{z}\int_0^z\frac{1+t}{1-t}dt\right]_{z=e^{i\theta}}=-\int_0^1\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}d\rho<0, \quad (1.9)$$

$$V(\theta)=\frac{dU(\theta)}{d\theta}=\int_0^1\frac{2\rho(1-\rho^2)\sin\theta}{(1-2\rho\cos\theta+\rho^2)^2}d\rho>0. \quad (1.10)$$

The function $w=S(z)$ maps $|z|=1$ onto an unbounded curve C, symmetric about the real axis. From (1.9) it follows that C lies in the left half-plane Re $w<0$. From (1.10) it follows that the imaginary part of $S(e^{i\theta})$ is positive for $0<\theta<\pi$ and that its real part is a strictly increasing function of θ. Thus $S(z)$ is both typically-real and univalent in $|z|<1$.

Moreover, $S(z)$ is starlike with respect to the origin in $|z|<1$. In other words,

$$\mathrm{Re}\,\frac{zS'(z)}{S(z)}>0 \quad (|z|<1). \quad (1.11)$$

Since $S(z)=\dfrac{2}{z}\displaystyle\int_0^z K(t)dt$, it follows that

$$zS'(z)+S(z)=2K(z)=2z(1-z)^{-2},$$

we have the relations

$$\frac{zS'(z)}{S(z)}+1=\frac{2z}{(1-z)^2S(z)}, \quad \frac{zS'(z)}{S(z)}-1=\frac{2z-2(1-z)^2S(z)}{(1-z)^2S(z)}.$$

But (1.11) is equivalent to

$$\left|\frac{zS'(z)}{S(z)}-1\right|<\left|\frac{zS'(z)}{S(z)}+1\right| \quad (|z|<1).$$

In other words, (1.11) is equivalent to

$$\left|3z-2-\frac{2}{z}(1-z)^2\log(1-z)\right|=\left|\sum_{n=1}^{\infty}\frac{4z^{n+1}}{n(n+1)(n+2)}\right|<|z|^2<|z|. \tag{1.12}$$

The inequality in (1.12) follows from the identity

$$4\sum_{n=1}^{\infty}\frac{1}{n(n+1)\ (n+2)}=1.$$

It is also obvious that (1.11) is satisfied for $z=0$. This completes the proof that $S(z)$ is univalent and starlike with respect to the origin for $|z|<1$.

2. An extremal problem and Its application

We obtain the following theorems and corollaries.

THEOREM 1. *Let $\mathcal{P}$ be the class of regular functions $P(z)$ in $|z|<1$ with $P(0)=1$ and* Re$P(z)>0$. *Let $F(w)$ denote a nonconstant function that is analytic in the convex domain D_0 which is the image of $|z|<1$ by the mapping*

$$w=\mu_0(z)=\frac{1}{z}\int_0^z\frac{1+t}{1-t}dt=-\frac{2}{z}\log(1-z)-1.$$

This domain D_0 lies the half-strip given by the inequalities $|\text{Im } w|<\pi$, Re$w>2\log 2-1$. *Then, for each $r<1$, the minimum*

$$m_F(r)=\min_{p\in P}\ \min_{|z|=r}\ \text{Re } F\left(\frac{1}{z}\int_0^z P(t)\ dt\right)$$

occurs for a function of the form $P(z)=(1+\varepsilon z)(1-\varepsilon z)^{-1}$, where ε is an arbitrary complex constant of absolute value 1, *and for no other functions.*

Proof. Let $\mu_0(z)$ be defined as in (1.6). Then

$$\mu_0(z)=\frac{1}{z}\int_0^z\frac{1+t}{1-t}dt=\int_0^1\frac{1+z\rho}{1-z\rho}d\rho=-\frac{2}{z}\log(1-z)-1.$$

R.M. Robinson [11] has pointed out that $\mu_0(z)$ is convex in $|z|<1$. However, this is also an immediate consequence of Lemma 1. Since $S(z)$ is starlike in $|z|<1$ and since

$$z\mu_0'(z)=S(z)=\frac{2}{1-z}+\frac{2}{z}\log(1-z),$$

it follows that $w=\mu_0(z)$ maps $|z|<1$ onto a convex domain D_0. This domain is symmetric about the real axis. We shall show that D_0 lies in the half-plane Re $w>2\log 2-1>0$ between the two straight lines $\mathrm{Im} w=\pm\pi$, a fact that is not immediately obvious. We let

$$\mu_0(e^{i\theta})=U_0(\theta)+iV_0(\theta).$$

Then, for $0<\theta\leqq\pi$,

$$U_0(\theta)=-(\cos\theta)\log\{2(1-\cos\theta)\}-1+(\pi-\theta)\sin\theta,$$
$$V_0(\theta)=(\sin\theta)\log\{2(1-\cos\theta)\}+(\pi-\theta)\cos\theta,$$
$$\frac{d}{d\theta}V_0(\theta)=-U_0(\theta). \tag{2.1}$$

From (1.7) and (1.9) we see that $U_0(\theta)=-U(\theta)>0$. From (2.1) it follows that $V_0'(\theta)<0$ for $0<\theta\leqq\pi$. Since $V_0(0)=\pi$ and $V_0'(\theta)<0$ for $0<\theta\leqq\pi$, it now follows that

$$|\mathrm{Im}\ \mu_0(z)|<\pi \quad (|z|<1).$$

Since $U_0(\pi)=2\log 2-1$ and $U_0(0)=+\infty$, and because $\mu_0(z)$ is univalent, convex and real on the real axis,

$$\mathrm{Re}\ \mu_0(z)>2\log 2-1 \qquad (|z|<1).$$

Since $P(z)\in\mathcal{P}$ has a Herglotz representation

$$P(z)=\int_0^{2\pi}\frac{1+ze^{i\phi}}{1-ze^{i\phi}}\,d\alpha(\phi) \qquad \left(\int_0^{2\pi}d\alpha(\phi)=1\right),$$

where $\alpha(\phi)$ is a nondecreasing function of ϕ in $[0, 2\pi]$, we can write

$$\mu(z)=\int\frac{1}{z}\int_0^z P(z)\,dt=\int_0^{2\pi}\left(\frac{1}{z}\int_0^z\frac{1+te^{i\phi}}{1-te^{i\phi}}dt\right)d\alpha\ (\phi)$$
$$=\int_0^{2\pi}\left(\int_0^1\frac{1+\rho ze^{i\phi}}{1-\rho ze^{i\phi}}d\rho\right)d\alpha(\phi)=\int_0^{2\pi}\eta_0(ze^{i\phi})\,d\alpha(\phi).$$

We see that if $|z|=r$, then $\mu(z)$ is an average (with a positive weight factor) values $\mu_0(ze^{i\phi})$, and hence it lies in the convex hull of the values of $\mu_0(z)$ for $|z|=r$. But since $\mu_0(z)$ is univalent and convex, it follows (as R. M. Robinson observed [11]) that $\mu(z)$ is subordinate to $\mu_0(z)$ for $|z|<1$. Because $\mu(0)=\mu_0(0)=1$, there exists a bounded function $\omega(z)$, regular in $|z|<1$, such that

$$\omega(0)=0,\quad |\omega(z)|\leqq|z|<1,\quad \mu(z)=\mu_0\{\omega(z)\}.$$

Hence, if $F(w)$ is an analytic function, regular in the convex domain D_0 that is the image of $|z|<1$ by the mapping $w=\mu_0(z)$, then

$$F(\mu(z))=F(\mu_0\{\omega(z)\}).$$

This states that $F(\mu(z))$ for $|z|=\mathrm{r}$ always lies within the set of values $F(\mu_0(z))$ for $|z|\leqq r$, it follows that

$$\min_{p\in\mathcal{P}}\min_{|z|=r}\ \mathrm{Re}\ F(\mu(z))=\min_{|z|=r}\ \mathrm{Re}\ F(\mu_0(z)). \qquad (2.2)$$

Thus $P_0(z)=(1+z)(1-z)^{-1}$ is an extremal function for the left member of (1.2). If $\varepsilon=e^{i\alpha}$, where α is real, then clearly $P_0(\varepsilon z)$ is also an extremal function, since the right-hand side of (2.2) is unchanged when z is replaced by εz. The only extremal functions are of the form $P_0(\varepsilon z)$, since by Schwarz's Lemma $|\omega(z)|<|z|$ unless $\omega(z)=e^{i\phi}z$. For if $P_1(z)$ is another extremal function, not of the form $P_0(\varepsilon z)$, then

$$\min_{|z|=r}\mathrm{Re}\,F\left(\frac{1}{z}\int_0^z P_1(t)dt\right)=\min_{|z|=r}\mathrm{Re}\ F(\mu_0(z)).$$

Then there exists an $\omega_1(z)$ $(|\omega_1(z)|\leqq|z|<1)$, regular in $|z|<1$, such that

$$\min_{|z|=r}\ \mathrm{Re}\ F(\mu_0\{\omega_1(z)\})=\min_{|z|=r}\ \mathrm{Re}\ F(\mu_0(z)),$$

which for nonconstant $F(z)$ is impossible by the minimum theorem, unless $|\omega_1(z)|=|z|$. If $\omega_1(z)=\varepsilon z$, where $|\varepsilon|=1$, then

$$\frac{1}{z}\int_0^z P_1(t)dt=\mu_0\omega_1(z))=\frac{1}{\varepsilon z}\int_0^{\varepsilon z}\frac{1+t}{1-t}dt=\frac{1}{z}\int_0^z\frac{1+\varepsilon\rho}{1-\varepsilon\rho}d\rho.$$

Thus, for all z and z_1 in $|z|<1$, we see that

$$\int_0^z\left(P_1(\rho)-\frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right)d\rho=0,\quad \int_0^{z_1}\left(P_1(\rho)-\frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right)d\rho=0.$$

For $z\neq z_1$, this gives the relation

$$\frac{1}{z-z_1}\int_{z_1}^{z}\left(P_1(\rho)-\frac{1+\varepsilon\rho}{1-\varepsilon\rho}\right)d\rho=0.$$

Letting $z_1\longrightarrow z$, we conclude that

$$P_1(z)=\frac{1+\varepsilon z}{1-\varepsilon z}=\mathrm{P}_0(\varepsilon z).$$

This completes the proof of Theorem 1.

THEOREM 2. *Let $\mathcal{P}$ be the class described in Theorem 1. Let z_1 be a complex number $(0<|z_1|<1)$. Let $F(w)$ denote a nonconstant function that is analytic in the convex domain D_0 defined as in Theorem 1. Then*

$$\min_{p\in\mathcal{P}}\mathrm{Re}\,\mathrm{F}\left(\frac{1}{z_1}\int_0^{z_1}P(z)dz\right)=\mathrm{Re}\,F\left(\frac{1}{z_1}\int_0^{z_1}\frac{1+\varepsilon z}{1-\varepsilon z}dz\right)$$

where ε depends on z_1 and F, and $|\varepsilon|=1$.

Proof. For $|z_1|=\mathrm{r}$, $0<\mathrm{r}<1$, we obtain, from Theorem 1,

$$\min_{p\in\mathcal{P}}\min_{|z|=r}\operatorname{Re}F(\mu(z))\leqq\operatorname{Re}F(\mu(z_1)). \tag{2.3}$$

If the left-hand side of (2. 3) is attained at the point z_0 ($|z_0|=r$) when

$$P(z)=P_0(z)=(1+z)(1-z)^{-1},$$

and if $z_0z_1^{-1}=\varepsilon$, then ε depends on F and z_1. Furthermore, equality occurs in (2. 3) if on the right-hand side of (2. 3) in the definition of

$$\mu(z_1)=\int_0^1 P(\rho z_1)d\rho,$$

$P(z)$ is replaced by $P_0(\varepsilon z)=P_0(z_0z_1^{-1}z)$. In this case, the right-hand side becomes Re $F(\mu_0(z_0))$. However, the left-hand side is also Re $F(\mu_0(z_0))$. Hence it follows from (2. 3) that

$$\min_{p\in\mathcal{P}}\operatorname{Re}F(\mu(z_1)) \text{ is attained for } P(z)=P_0(\varepsilon z).$$

This completes the proof of Theorem 2.

Applications to the class $\mathcal{R}_\alpha$ of analytic function

$$f(z)=e^{i\alpha}z+a_2z^2+\cdots+a_nz^n+\cdots$$

that are regular and satisfy the condition Re $f'(z)>0$ in $|z|<1$ will be given by the following two corollaries;

COROLLARY 1. *Let $\mathcal{R}_\alpha$ be the class of regular functions $f(z)$ defined in $|z|<1$ as in* (1. 3), *so that* Re$f'(z)>0$ *in* $|z|<1$. *Let $G(w)$ denote a nonconstant analytic function in the convex D_α that is the image of $|z|<1$ by the mapping*

$$w=-e^{-i\alpha}-\frac{2}{z}(\cos\alpha)\log(1-z).$$

Then, for each fixed $r<1$, the minimum

$$\min_{f\in\mathcal{R}_\alpha}\min_{|z|=r}\operatorname{Re}G\left(\frac{f(z)}{z}\right)$$

occurs for a function of the form

$$f(z)=-e^{-i\alpha}z-\frac{2}{\varepsilon}(\cos\alpha)\log(1-\varepsilon z),$$

where ε is an arbitrary complex constant of absolute value 1, *and for no other functions.*

Proof. Since Re $f'(z)>0$, we may write

$$f'(z)=(\cos\alpha)\ P(z)+i\sin\alpha,\quad (\cos\alpha>0,\ P(z)\in\mathcal{P}),$$

$$\frac{f(z)}{z}=i\sin\alpha+\frac{\cos\alpha}{z}\int_0^z P(t)dt=i\sin\alpha+(\cos\alpha)\mu(z).$$

When

$$P(z)=P_0(\varepsilon z)=\frac{1+\varepsilon z}{1-\varepsilon z}\quad(|z|=1),$$

then

$$f(z)=f_0(z)=-e^{-i\alpha}z-\frac{2}{\varepsilon}(\cos\alpha)\log(1-\varepsilon z).$$

The function

$$w=w(z)=\frac{\varepsilon}{z}f_0(\bar{\varepsilon}z)=-e^{-i\alpha}-\frac{2}{z}(\cos\alpha)\log(1-z)$$

is convex in $|z|<1$, since $zw'(z)=(\cos\alpha)\,S(z)$ is starlike in $|z|<1$. Thus Corollary 1 follows immediately from Theorem 1.

COROLLARY 2. *Let $\mathcal{R}_\alpha$ be the class of regular functions $f(z)$ defined in $|z|<1$ as in* (1.3), *so that $\mathrm{Re} f'(z)>0$ in $|z|<1$. Let $G(w)$ denote a function that is analytic in the right half-plane* $\mathrm{Re}w>0$. *Then for each $r<1$*

$$\min_{f\in\mathcal{R}_\alpha}\min_{|z|=r}\mathrm{Re}\,G(f'(z))=\min_{|z|=r}\mathrm{Re}\,G\left((\cos\alpha)\frac{1+z}{1-z}+i\sin\alpha\right).$$

Proof. This corollary is an easy consequence of Theorem 2 of Robertson's paper [10], which states that if $F(w)$ is analytic in the half-plane Re $w>0$, then for each $r<1$

$$\min_{p\in\mathcal{P}}\min_{|z|=r}\mathrm{Re}F\,(P(z))=\min_{|z|=r}\mathrm{Re}\,F\left(\frac{1+z}{1-z}\right).\tag{2.4}$$

So corollary 2 follows from (2.4) if we take

$$F(P(z))=G((\cos\alpha)\,P(z)+i\sin\alpha)=G(f'(z)).$$

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