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PROJECTIVE REPRESENTATIONS OF SOME FINITE GROUPS

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1. Introduction

Let G be a finite group, and let K be an algebraically closed field of characteristic zero with its multiplicative group $K^*=K-\{0\}$. A mapping $T:G\longrightarrow GL_n(K)$ of G into the general linear group $GL_n(K)$ is called a *projective representation* of G of degree n over K if

$$T(g)T(h)=\alpha(g,h)T(gh),\ \alpha(g,h)\in K^*$$

holds for all $g,\ h\in G$. The function $\alpha:G\times G\longrightarrow K^*$ is called the *factor set* of T. We say that T is *irreducible* if the vector space $V=K^n$ has no nontrivial proper subspace invariant under all $T(g),\ g\in G$.

Let $T:G\longrightarrow GL_n(K)$ and $S:G\longrightarrow GL_n(K)$ be projective representations of G with factor sets α and β, respectively. We say that T and S are (*projectively*) *equivalent* if there exists a nonsingular matrix $P\in GL_n(K)$ and a function $c:G\longrightarrow K^*$ such that

$$S(g)=c(g)P^{-1}T(g)P,\quad g\in G.$$

In this case, α and β are *equivalent*. That is, the following holds:

$$\beta(g,h)=\alpha(g,h)c(g)c(h)c(gh)^{-1},\ g,h\in G.$$

If there exists a nonsingular matrix $P\in GL_n(K)$ such that

$$S(g)=P^{-1}T(g)P,\qquad g\in G,$$

then T and S are said to be *linearly equivalent*. Linearly equivalent projective representations have the same factor set.

The purpose of this paper is to explicitly determine all the irreducible projective representations of some finite groups. The following is our main Theorem.

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THEOREM *Let G be a finite group with two generators x and y defined by*

$$G=\langle x,\ y \mid x^n=1,\ y^p=1,\ y^{-1}xy=x^r\rangle,$$

where

(i) $(r, n)=1$, p *is a prime,*
(ii) $(r-1,\ nd)=d$ *with* $d=(r-1,\ n)$, *and*
(iii) $r^p\equiv 1 \pmod{nd}$.

Let K be an algebraically closed field of characteristic zero and ξ *a primitive n-th root of unity in K. Then d is equal to* 1 *or* p.

If $d=1$, *then we have* $H^2(G, K^*)=\{1\}$.

If $d=p$, *then the following hold.*

(1) $H^2(G,\ K^*)$ *is a cyclic group of order* p.

(2) *For any factor set* β *of G over K which is not equivalent to the trivial factor set,*

$$K^\beta[G]\cong M_1\oplus\cdots\oplus M_{\frac{n}{p}},\quad M_1=\cdots=M_{\frac{n}{p}}=Mat_p(K).$$

(3) *Let* $\alpha: G\times G\longrightarrow K^*$ *be a factor set of G such that*

$$\alpha(y^jx^i, x^k)=\alpha(y^j, y^i)=1,$$
$$\alpha(y^jx^i, y^lx^k)=\xi^{(1+r+r^2+\cdots+r^{l-1})i}$$

for all integers i, j, k *and* $l\geq 1$. *Then*

$$H^2(G, K^*)=\langle\{\alpha\}\rangle=\{\{1\},\ \{\alpha\},\ \{\alpha^2\},\cdots,\ \{\alpha^{p-1}\}\}.$$

For each factor set α^k *with* $1\leq k\leq p-1$, *there are exactly* $\frac{n}{p}$ *linearly inequivalent irreducible projective representations of G over K with factor set* α^k. *They are projective representations* $T_{ki}: G\longrightarrow GL_p(K)$ *of degree* p *defined by*

$$T_{ki}(x)=\text{diag}\ \{\xi^{k+i},\ \xi^{k(1+r)+ri},\ \xi^{k(1+r+r^2)+r^2i},\ \ldots,\ \xi^{k(1+r+\cdots+r^{p-1})+r^{p-1}i}\}$$

$$T_{ki}(y)=\begin{pmatrix}0&0&\cdots&0&1\\1&0&&&0\\0&1&&&0\\&&\ddots&&\\0&0&\cdots&1&0\end{pmatrix},\quad T_{ki}(y^jx^l)=T_{ki}(y)^jT_{ki}(x)^l.$$

Here, for each k, projective representations T_{ki} *and* T_{kj} *are linearly equivalent if and only if* $j\equiv k(r+r^2+\cdots+r^l)+r^li \pmod{n}$ *for some* l *with* $0\leq l\leq p-1$.

The proof of *Theorem* will be given in section 3. In fact, all the irreducible ordinary representations of the group G over K are wellknown and they can be found in [2, §47]. On the other hand, if α and β are equivalent factor sets of G over K satisfying

$$\beta(g, h)=\alpha(g, h)c(g)c(h)c(gh)^{-1}, \qquad g, h\in G,$$

for some function $c : G \longrightarrow K^*$, then

$$T \longrightarrow cT$$

is a one-to-one correspondence between the projective representations of G with factor set α and those with factor set β, which preserves irreducibility and linear equivalence. Therefore, our *Theorem* gives all the irreducible projective representations of G over K.

The dihedral group

$$D_n=\langle x, y \mid x^n=1, y^2=1, y^{-1}xy=x^{-1}\rangle$$

of order $2n$, $n\geq 3$, satisfies the conditions in *Theorem*. Thus if n is odd then $H^2(D_n, K^*)=\{1\}$, and if n is even then $H^2(D_n, K^*)$ is a cyclic group of order 2. And every irrducible projective representations of D_n over K are determined.

The notation and terminology in this paper are standard. For any finite group G, denote the *order*, the *center* and the *commutator subgroup* of G by $|G|$, $Z(G)$, and $[G, G]$, respectively. The full matrix ring over K of degree n will be denoted by $Mat_n(K)$.

The equivalence class containing a factor set α of G will be denoted by $\{\alpha\}$. The set $H^2(G, K^*)$ of all equivalence classes of factor sets of G over K forms an abelian group under the multiplication defined by $\{\alpha\}\{\beta\}=\{\alpha\beta\}$. This group is called the *Schur multiplier* of G, and it is actually the second cohomology group where K^* is a trivial G-module.

Let $K^\alpha[G]$ denote the twisted group algebra of G over K with respect to the factor set α. Then there is a one-to-one correspondence between modules over $K^\alpha[G]$ and the projective representations of G over K with factor set α. And the isomorphism of modules corresponds to the linear equivalence of projective representations. Note that if α is the trivial factor set then $K^\alpha[G]$ is the ordinary group algebra $K[G]$ and a projective representation of G with factor set α is an ordinary representation of G.

2. Necessary Lemmas

The following two lemmas will be used in the proof of *Theorem*.

(2.1) *Let n, r and p be integers such that*

(i) $(r, n)=1$, *n is positive, p is a prime,*

(ii) $(r-1,\ nd)=d$ *with* $d=(r-1, n)$, *and*

(iii) $r^p\equiv 1 \pmod{nd}$.

Then the following hold.

(1) *If* $ri\equiv i \pmod{nd}$, *then* $i\equiv 0 \pmod{n}$.

If $r^i\equiv 1 \pmod{nd}$, $i\geq 1$, then $1+r+\cdots+r^{i-1}\equiv 0 \pmod{n}$.

(2) *We have*

$r^p\equiv 1 \pmod{n}$ *and* $1+r+\cdots+r^{p-1}\equiv 0 \pmod{n}$.

(3) *If i and j are positive integers such that* $i\equiv j \pmod{p}$, *then*

$$1+r+\cdots+r^{i-1}\equiv 1+r+\cdots+r^{j-1} \pmod{n}.$$

(4) *The integer d is equal to either 1 or p. Moreover, $d=1$ if and only if $p\nmid n$, and $d=p$ if and only if $p\mid n$.*

Proof. The assertions (1) and (2) follow from the facts that

$$\left(\frac{r-1}{d},\ n\right)=1 \text{ and } r^i-1=(r-1)(1+r+\cdots+r^{i-1}).$$

And the assertion (3) follows from (2).

Since $(r-1,\ n)=d$, we have $r\equiv 1 \pmod{d}$ and $n\equiv 0 \pmod{d}$. By (2) this implies that

$$0\equiv 1+r+\cdots+r^{p-1}\equiv 1+1+\cdots+1=p \pmod{d}.$$

Therefore, d is equal to either 1 or p.

If $d=p$, then $p\mid n$. Conversely, assume that $p\mid n$. Then $r^p\equiv 1 \pmod{p}$ by (2), and so $r\equiv r^p\equiv 1 \pmod{p}$. Hence p divides d, and so we have $d=p$. Thus the assertion (4) holds.

(2.2) *Let G be a group with two generators x and y defined by*

$$G=\langle x,\ y\mid x^n=1,\ y^p=1,\ y^{-1}xy=x^r\rangle,$$

where $(r, n)=1$, *p is a prime, and* $r^p\equiv 1 \pmod{n}$.

Set $d=(r-1, n)$, *and let K be an algebraically closed field of characteristic zero. Then the following hold.*

(1) *G is a group of order np, and*

$$[G, G]=\langle x^{r-1}\rangle=\langle x^d\rangle,\ Z(G)=\langle x^{\frac{n}{d}}\rangle.$$

(2) *We have*

$$K[G]\cong K_1\oplus\cdots\oplus K_{pd}\oplus M_1\oplus\cdots\oplus M_m,$$

where

$$m=\frac{n-d}{p},\quad K_1=\cdots K_{pd}=K,\quad M_1=\cdots=M_m=Mat_p(K).$$

Proof. The proof can be found in [2, p. 338].

3. Proof of Theorem

In this section we will prove our *Theorem* by a series of propositions.

Note that G is a well defined group, and it is a semidirect product of a normal subgroup $\langle x\rangle$ by a subgroup $\langle y\rangle$. Thus every element g of G can be expressed uniquely in the form

$$g=y^jx^i,\quad 0\leq i\leq n-1,\quad 0\leq j\leq p-1.$$

(3.1) *The integer d is equal to either 1 or p. Moreover, $d=1$ if and only if $d\nmid n$, and $d=p$ if and only if $p\mid n$.*

Proof. This follows from (2.1).

(3.2) *If $d=1$, then $H^2(G,K^*)=\{1\}$.*
Proof. If $d=1$, then $p\nmid n$. Hence every Sylow subgroups of G is cyclic, which implies that $H^2(G,K^*)=\{1\}$.

(3.3) *Assume that $d=p$. Then $H^2(G,K^*)$ is a cyclic group of order p, and*

$$G_1=\langle x_1,y_1\mid x_1^{np}=1,\ y_1^p=1,\ y_1^{-1}x_1y_1=x_1^r\rangle$$

is a representation group of G.

Proof. Since G is generated by two elements and three defining relations, $H^2(G,K^*)$ must be cyclic. On the other hand, G has a normal subgroup $\langle x\rangle$ of index p. Hence $|H^2(G,K^*)|\leq p$.

Since $p\mid n$ and $(r,n)=1$, it follows that $(r,np)=1$. And we have $r^p\equiv 1 \pmod{np}$. Hence G_1 is well defined. By (2.2), we have

$$[G_1,G_1]=\langle x_1^{r-1}\rangle=\langle x_1^p\rangle,\quad Z(G_1)=\langle x_1^n\rangle.$$

Thus $\langle x_1^n\rangle=[G_1,G_1]\cap Z(G_1)$ and $G_1/\langle x_1^n\rangle\cong G$. Hence it follows that $|H^2(G,K^*)|\geq p$. Therefore, $H^2(G,K^*)$ is cyclic of order p, and G_1 is a representation group of G.

(3.4) *Assume that $d=p$, and let $\beta : G\times G \longrightarrow K^*$ be a factor set of G which is not equivalent to the trivial factor set. Then*

$$K^{\beta}[G]\cong M_1\oplus \cdots \oplus M_{\frac{n}{p}}, \quad \text{where } M_1=\cdots=M_{\frac{n}{p}}=Mat_p(K).$$

And every irreducible projective representation of G over K with factor set β is of degree p, and every projective representation of G over K of degree p with factor set β is irreducible.

Proof. Since G_1 is a representation group of G, we have

$$K[G_1]\cong \oplus_{\{\alpha\}} K^{\alpha}[G],$$

where the sum runs over all elements $\{\alpha\}$ in $H^2(G, K^*)$. On the other hand, it follows from (2.2) that

$$K[G_1]\cong K_1\oplus \cdots \oplus K_{p^2}\oplus M_1\oplus \cdots \oplus M_{n-1}$$

and

$$K[G]\cong K_1\oplus \cdots \oplus K_{p^2}\oplus M_1\oplus \cdots \oplus M_{\frac{n}{p}-1}$$

where $K_1=\cdots=K_{p^2}=K$ and $M_1=\cdots=M_{n-1}=Mat_p(K)$. Since β is not equivalent to the trivial factor set, $K^{\beta}[G]$ not isomorphic to $K[G]$. Hence it follows that

$$K^{\beta}[G]\cong M_1\oplus \cdots \oplus M_{\frac{n}{p}}.$$

Thus the assertions hold.

(3.5) *Assume that $d=p$. Let $\beta : G\times G \longrightarrow K^*$ be a factor set of G such that*

$$\beta(y^j x^i,\ x^k)=\beta(y^j,\ y^i)=1$$

for all integers i, j and k, and set $\beta(x, y)=B$.

Then B is an n-th root of unity in K, and

$$\beta(y^j x^i,\ y^l x^k)=B^{(1+r+\cdots+r^{l-1})i}$$

for all integers i, j, k and $l\geq 1$. Moreover, β is equivalent to the trivial factor set if and only if $B\in\langle\xi^p\rangle$.

Proof. We use the identity

$$\beta(g, hk)\beta(h, k)=\beta(g, h)\beta(gh, k), \qquad g, h, k\in G.$$

First we can show that

$$\beta(y^j x^i,\ y^l x^k)=\beta(y^j x^i,\ y^l),$$

and, by induction on $l\geq 1$, we can show that

$$\beta(y^j x^i, y^l)=B^{(1+r+\cdots+r^{l-1})i}.$$

Since $x^n=1$, we must have $1=\beta(x^n, y)=B^n$ and so B is an n-th root of unity in K. Furthermore, β is a well defined factor set of G, by (2) and (3) of (2.1).

Now suppose that β is equivalent to the trivial factor set 1. Then there exists a function $c: G \longrightarrow K^*$ such that

$$\beta(g, h)=c(g)c(h)c(gh)^{-1}, \quad g, h\in G.$$

Since $\beta(x^i, x^j)=1$ for all i and j, the mapping $c|_{\langle x\rangle} : \langle x\rangle \longrightarrow K^*$ is a group-homomorphism. Hence we have

$$c(x^i)=c(x)^i, \quad c(x)^n=c(1)=1.$$

From $\beta(y, x^i)=1$ it follows that $c(yx^i)=c(y)c(x)^i$. Hence

$$B=\beta(x, y)=c(x)c(y)c(yx^r)^{-1}=c(x)^{1-r},$$

and so B is contained in $\langle\xi^{1-r}\rangle=\langle\xi^p\rangle$.

Conversely, suppose that B is contained in $\langle\xi^p\rangle$ and let γ be an element of $\langle\xi\rangle$ such that $B=\gamma^{1-r}$. Define a function $c: G\longrightarrow K^*$ by $c(y^jx^i)=\gamma^i$ for all integers i and j. Then it is easy to see that $\beta(g, h)=c(g)c(h)c(gh)^{-1}$ holds for all $g, h\in G$. Thus β is equivalent to 1.

(3.6) *Theorem holds.*

Proof. By (3.2), if $d=1$ then $H^2(G, K^*)=\{1\}$.

Now assume that $d=p$. By (3.3) and (3.4), the assertions (1) and (2) of *Theorem* hold. Let $\alpha: G\times G\longrightarrow K^*$ be a factor set such that

$$\alpha(y^jx^i, x^k)=\alpha(y^j, y^i)=1$$

and

$$\alpha(y^jx^i, y^lx^k)=\xi^{(1+r+\cdots+r^{l-1})i}$$

for all integers i, j, k and $l\geq 1$. Then $\alpha(x, y)=\xi$, and so it follows from (3.5) that α^k is equivalent to the trivial factor set if and only if $p|k$. This implies that $\langle\{\alpha\}\rangle$ is a subgroup of $H^2(G, K^*)$ of order p. Hence $H^2(G, K^*)=\langle\{\alpha\}\rangle$.

Let k be an integer such that $1\leq k\leq p-1$. Then the factor set α^k is not equivalent to the trivial factor set, and so (3.4) holds for $\beta=\alpha^k$. Hence there are exactly $\frac{n}{p}$ linearly inequivalent irreducible projective representations of G over K with factor set α^k.

Define a mapping $T_{ki}: G\longrightarrow GL_p(K)$ as in *Theorem*. Then it is easy to see that T_{ki} is a projective representation of G over K with factor set α^k. By (3.4) every T_{ki} is irreducible.

Suppose that T_{ki} and T_{kj} are linearly equivalent. Then $T_{ki}(x)$ and $T_{kj}(x)$ must have the same eigenvalues, and so ξ^{k+j} must be equal to $\xi^{k(1+r+\cdots+r^l)+r^l i}$ for some integer l with $0 \leq l \leq p-1$. This implies that

$$j \equiv k(r+r^2+\cdots+r^l)+r^l i \pmod{n}$$

Conversely, assume that the above congruence holds. Then it is easy to see that there exists a permutation matrix $P \in GL_p(K)$ such that

$$T_{kj}(g)=P^{-1}T_{ki}(g)P, \quad g \in G.$$

Hence T_{ki} and T_{kj} are linearly equivalent.

This completes the proof of *Theorem*.

References

1. Beyl, F.R. and Tappe, J., *Group extensions, Representations, and the Schur multiplicator*, Lecture Notes in Math., Vol. **958**, Springer-Verlag, New York, 1982
2. Curtis, W.C. and Reiner, I., *Representation theory of finite groups and associative algebras*, Pure Appl. Math., Vol. **11**, Interscience, New York, 1962
3. Gaschuetz, W., Neubueser, J., and Yen, T., *Ueber den Multiplicator von-p-Gruppen*, Math. Z. **100** (1967), 93-96.
4. Huppert, B., *Endliche Gruppen* I, Grundlehren der Math. Wissenschaften, Band **134**, Springer-Verlag, Berlin, 1967.
5. Schur, I., *Untersuchungen ueber die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **132** (1907), 155-250.

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