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ON NEW SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS*

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of analytic and univalent functions in the unit disk $\mathcal{U}$. Then a function $f(z)$ of $\mathcal{S}$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]>\alpha \qquad (z\in\mathcal{U}) \tag{1.2}$$

for some α $(0\leqq\alpha<1)$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order α.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order α if and only if

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\alpha \qquad (z\in\mathcal{U}) \tag{1.3}$$

for some $\alpha(0\leqq\alpha<1)$. We denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order α. We note that $f(z)\in\mathcal{K}(\alpha)$ if and only if $zf'(z)\in\mathcal{S}^*(\alpha)$, and that

$$\mathcal{S}^*(\alpha)\subset\mathcal{S}^*(0)\equiv\mathcal{S}^*,\ \ \mathcal{K}(\alpha)\subset\mathcal{K}(0)\equiv\mathcal{K},\ \text{and}\ \mathcal{K}(\alpha)\subset\mathcal{S}^*(\alpha)$$

for $0\leqq\alpha<1$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [10], and were studied subsequently by Schild [13], MacGregor [4], Pinchuk

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[9], and others.

Many essentially equivalent definitions of the fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [2, Chapter 13], [6], [7], [11], [12], [15, p. 28 et seq.], [17], and [18]). We find it to be convenient to recall here the following definitions which were used recently by Owa [8] (and by Srivastava and Owa [16]).

DEFINITION 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta, \tag{1.4}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta, \tag{1.5}$$

where $0\leqq\lambda<1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{1.6}$$

where $0\leqq\lambda<1$, $n\in\mathcal{N}_0=\{0,1,2,\dots\}$.

Let $\mathcal{S}^*(\alpha,\lambda)$ denote the class of all functions $f(z)$ in $\mathcal{S}$ satisfying the inequality

$$\mathrm{Re}\left[\frac{\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)}{f(z)}\right]>\alpha \qquad (z\in\mathcal{U}) \tag{1.7}$$

for $\lambda<1$ and $0\leqq\alpha<1$. Also let $\mathcal{K}(\alpha,\lambda)$ denote the class of all functions $f(z)$ in $\mathcal{S}$ such that

$$\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)\in\mathcal{S}^*(\alpha,\lambda) \tag{1.8}$$

for $\lambda<1$ and $0\leqq\alpha<1$. Clearly,

$$\mathcal{S}^*(\alpha, 0)=\mathcal{S}^*(\alpha) \text{ and } \mathcal{K}(\alpha, 0)=\mathcal{K}(\alpha)$$

where we have set $\lambda=0$. Thus $\mathcal{S}^*(\alpha, \lambda)$ and $\mathcal{K}(\alpha, \lambda)$ are generalizations of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively.

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of all functions of the form

$$(1.9) \qquad f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0).$$

Denote by $\mathcal{T}^*(\alpha, \lambda)$ and $\mathcal{C}(\alpha, \lambda)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(\alpha, \lambda)$ and $\mathcal{K}(\alpha, \lambda)$ with $\mathcal{T}$, that is,

$$(1.10) \qquad \mathcal{T}^*(\alpha, \lambda)=\mathcal{S}^*(\alpha, \lambda) \cap \mathcal{T}$$

and

$$(1.11) \qquad \mathcal{C}(\alpha, \lambda)=\mathcal{K}(\alpha, \lambda) \cap \mathcal{T}.$$

The special cases of the classes $\mathcal{T}^*(\alpha, \lambda)$, and $\mathcal{C}(\alpha, \lambda)$, when $\lambda=0$, were studied by Silverman [14]. Thus the classes $\mathcal{T}^*(\alpha, \lambda)$ and $\mathcal{C}(\alpha, \lambda)$ provide interesting generalizations of the classes considered by Silverman [14].

In this paper, we first prove several interesting results for functions belonging to the general classes $\mathcal{T}^*(\alpha, \lambda)$ and $\mathcal{C}(\alpha, \lambda)$. We then introduce the subclasses $\mathcal{T}^*(\alpha, \lambda, \kappa)$ and $\mathcal{C}(\alpha, \lambda, \kappa)$ of functions with fixed finitely many coefficients. And, finally, we consider the classes of univalent functions with quasiconformal extension.

2. Coefficient Inequalities

THEOREM 1. *Let the function $f(z)$ be defined by* (1.1). *If*

$$(2.1) \qquad \sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}|a_n| \leqq 1-\alpha$$

for $\lambda<1$ and $0 \leqq \alpha<1$, then $f(z) \in \mathcal{S}^*(\alpha, \lambda)$. The result (2.1) *is sharp.*

Proof We need only prove that (2.1) implies (1.7). In order to prove (1.7), it suffices to show that

$$(2.2) \qquad \left|\frac{\Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)}{f(z)}-1\right|<1-\alpha \qquad (z \in \mathcal{U}).$$

Note that

$$(2.3) \qquad D_z^{1+\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)} z^{-\lambda}+\sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda)} a_n z^{n-1-\lambda},$$

which readily yields

$$(2.4)\qquad \left|\frac{\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)}{f(z)}-1\right|$$
$$=\left|\frac{\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-1\right\}a_n z^n}{z-\sum_{n=2}^{\infty}a_n z^n}\right|$$
$$\leqq\frac{\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-1\right\}a_n}{1-\sum_{n=2}^{\infty}a_n}$$
$$\leqq 1-\alpha,$$

provided that

$$(2.5)\qquad \sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-1\right\}a_n\leqq(1-\alpha)\left[1-\sum_{n=2}^{\infty}a_n\right].$$

We note that (2.5) is equivalent to (2.1). Further, the result (2.1) is sharp for the functions

$$(2.6)\qquad f(z)=z+\left[(1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n\qquad(n\geqq 2).$$

Thus we complete the proof of Theorem 1.

REMARK 1. Letting $\lambda=0$ in Theorem 1, we obtain the corresponding result for $\mathcal{S}^*(\alpha)$ due to Silverman [14].

THEOREM 2. *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$(2.7)\qquad \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}|a_n|\leqq 1-\alpha$$

for $\lambda<1$ *and* $0\leqq\alpha<1$, *then* $f(z)\in\mathcal{K}(\alpha,\lambda)$. *The result* (2.7) *is sharp.*

Proof. Note that $f(z)\in\mathcal{K}(\alpha,\lambda)$ if and only if

$$\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)\in\mathcal{S}^*(\alpha,\lambda).$$

Therefore, on replacing a_n by $[\Gamma(n+1)\Gamma(1-\lambda)/\Gamma(n-\lambda)]a_n$ in Theorem 1, we have Theorem 2. Further, the result (2.7) is sharp for the functions

$$(2.8)\qquad f(z)=z+\left[(1-\alpha)\Big/\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n\qquad(n\geqq 2).$$

REMARK 2. Letting $\lambda=0$ in Theorem 2, we have the corresponding result for $\mathscr{K}(\alpha)$ given by Silverman [14].

THEOREM 3. *Let the function $f(z)$ be defined by* (1.9). *Then $f(z)$ belongs to the class $\mathscr{T}^*(\alpha,\lambda)$ if and only if*

$$\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq 1-\alpha \tag{2.9}$$

for $\lambda<1$ and $0\leqq\alpha<1$. The result (2.9) *is sharp.*

Proof. By means of Theorem 1, (2.9) implies that $f(z)\in\mathscr{T}^*(\alpha,\lambda)$. Suppose that $f(z)$ is in the class $\mathscr{T}^*(\alpha,\lambda)$. Then

$$\begin{aligned}(2.10)\qquad &\operatorname{Re}\left[\frac{\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)}{f(z)}\right]\\ &=\operatorname{Re}\left(\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}a_nz^{n-1}}{1-\sum_{n=2}^{\infty}a_nz^{n-1}}\right)\\ &>\alpha\end{aligned}$$

for $\lambda<1$, $0\leqq\alpha<1$, and $z\in\mathscr{U}$. Choose values of z on the real axis so that $\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)/f(z)$ is real. Upon clearing the denominator in (2.10) and letting $z\to1^-$, we obtain

$$1-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}a_n\geqq\alpha\left[1-\sum_{n=2}^{\infty}a_n\right] \tag{2.11}$$

which implies (2.9). The result (2.9) is sharp for the functions

$$f(z)=z-\left[(1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n\qquad(n\geqq2). \tag{2.12}$$

This completes the proof of Theorem 3.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.9) *belong to the class $\mathscr{T}^*(\alpha,\lambda)$. Then*

$$a_n\leqq(1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\qquad(n\geqq2). \tag{2.13}$$

The result (2.13) *is sharp for the functions $f(z)$ given by* (2.12).

Next we have

THEOREM 4. *Let the function $f(z)$ be defined by* (1.9). *Then $f(z)$ belongs to the class $\mathscr{C}(\alpha,\lambda)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq1-\alpha \tag{2.14}$$

for $\lambda<1$ *and* $0\leqq\alpha<1$. *The result* (2.14) *is sharp for the functions*

$$f(z)=z-\left[(1-\alpha)\Big/\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n \quad (n\geqq 2). \tag{2.15}$$

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.9) *belong to the class* $\mathscr{C}(\alpha,\lambda)$. *Then*

$$a_n\leqq(1-\alpha)\Big/\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\} \quad (n\geqq 2). \tag{2.16}$$

The result (2.16) *is sharp for the functions* $f(z)$ *given by* (2.15).

3. Distortion Theorems

By virtue of the coefficient inequalities, we can prove the following distortion theorems for functions $f(z)$ belonging to the classes $\mathscr{T}^*(\alpha,\lambda)$ and $\mathscr{C}(\alpha,\lambda)$.

LEMMA 1. *The class* $\mathscr{T}^*(\alpha,\lambda)$ *is closed under linear combinations.*

Proof. Let the functions

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \quad (i=1,2;\ a_{n,i}\geqq 0) \tag{3.1}$$

be in the class $\mathscr{T}^*(\alpha,\lambda)$. Then we need only prove that the function

$$\delta f_1(z)+(1-\delta)f_2(z) \quad (0\leqq\delta\leqq 1)$$

is also in the class $\mathscr{T}^*(\alpha,\lambda)$. In fact, we have

$$\delta f_1(z)+(1-\delta)f_2(z) =z-\sum_{n=2}^{\infty}[\delta a_{n,1}+(1-\delta)a_{n,2}]z^n. \tag{3.2}$$

Hence, with the aid of Theorem 3, we have

$$\begin{aligned}\sum_{n=2}^{\infty}&\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\left[\delta a_{n,1}+(1-\delta)a_{n,2}\right]\\ &=\delta\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_{n,1}\\ &\quad+(1-\delta)\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_{n,2}\\ &\leqq\delta(1-\alpha)+(1-\delta)(1-\alpha)\\ &=1-\alpha,\end{aligned} \tag{3.3}$$

which completes the proof of Lemma 1.

By means of Lemma 1, we know that $\mathscr{T}^*(\alpha,\lambda)$ is convex, and hence further that $\mathscr{T}^*(\alpha,\lambda)$ has some extreme points.

LEMMA 2. *Let*

$$f_1(z)=z \tag{3.4}$$

and

$$f_n(z)=z-\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}z^n \qquad (n\geqq 2). \tag{3.5}$$

Then $f(z)$ is in the class $\mathscr{T}^*(\alpha,\lambda)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{3.6}$$

where $\lambda_n\geqq 0$ $(n\geqq 1)$ and

$$\sum_{n=1}^{\infty}\lambda_n=1. \tag{3.7}$$

Proof. We assume that $f(z)$ has the form (3.6). Then

$$f(z)=z-\sum_{n=2}^{\infty}\frac{(1-\alpha)\lambda_n}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}z^n. \tag{3.8}$$

Consequently, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\cdot\frac{(1-\alpha)\lambda_n}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha} &=(1-\alpha)\sum_{n=2}^{\infty}\lambda_n\\ &=(1-\alpha)(1-\lambda_1)\\ &\leqq 1-\alpha,\end{aligned} \tag{3.9}$$

which implies that $f(z)\in\mathscr{T}^*(\alpha,\lambda)$.

For the converse, we assume that $f(z)$ is in the class $\mathscr{T}^*(\alpha,\lambda)$. Then, by setting

$$\lambda_n=\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n/(1-\alpha) \qquad (n\geqq 2) \tag{3.10}$$

and

(3.11) $$\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n,$$

we have (3.6).

In view of Lemma 2, we have

THEOREM 5. *The extreme points of* $\mathscr{T}^*(\alpha, \lambda)$ *are the functions* $f_n(z)$ $(n\geqq 1)$ *given by* (3.4) *and* (3.5).

Similarly, we have

THEOREM 6. *The extreme points of* $\mathscr{C}(\alpha, \lambda)$ *are the functions* $f_n(z)$ $(n\geqq 1)$ *given by* (3.4) *and*

(3.12) $$f_n(z)=z-\frac{1-\alpha}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}}z^n \quad (n\geqq 2).$$

THEOREM 7. *Let the function* $f(z)$ *defined by* (1.9) *belong to the class* $\mathscr{T}^*(\alpha, \lambda)$. *Then, if* $-1\leqq\lambda<1$

(3.13) $$|f(z)|\geqq|z|-\frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|^2$$

and

(3.14) $$|f(z)|\leqq|z|+\frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|^2$$

for $z\in\mathscr{U}$. *Furthermore, if* $0\leqq\lambda<1$,

(3.15) $$|f'(z)|\geqq 1-\frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|$$

and

(3.16) $$|f'(z)|\leqq 1+\frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|$$

for $z\in\mathscr{U}$. *The bounds* (3.13) *to* (3.16) *are sharp.*

Proof. Note that the extremal function is one of the extreme points. Therefore, we have

(3.17) $$|f(z)|\geqq|z|-\max_{n\geqq 2}\left\{\frac{1-\alpha}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}|z|^n\right\}$$

and

$$(3.18)\qquad |f(z)|\leqq|z|+\max_{n\geqq 2}\left\{\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}|z|^n\right\}.$$

Now define

$$(3.19)\qquad \mathcal{E}(\lambda,\alpha,n,|z|)=\frac{\Gamma(n-\lambda)|z|^n}{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)}$$

for $-1\leqq\lambda<1$, $0\leqq\alpha<1$, $n\geqq 2$, and $z\in\mathcal{U}$. If

$$(3.20)\qquad \mathcal{E}(\lambda,\alpha,n,|z|)\geqq\mathcal{E}(\lambda,\alpha,n+1,|z|)$$

for $|z|\neq 0$, then we have the first half of the theorem.

Set

$$(3.21)\qquad \mathcal{E}_1(\lambda,\alpha,n,|z|)=\Gamma(n+1)\Gamma(1-\lambda)\{(n+1)-(n-\lambda)|z|\}\\ -\alpha\Gamma(n+1-\lambda)(1-|z|).$$

If $\mathcal{E}_1(\lambda,\alpha,n,|z|)\geqq 0$, then we have (3.20). We note that $\mathcal{E}_1(\lambda,\alpha,n,|z|)$ is a decreasing function of α. This implies that

$$(3.22)\qquad \mathcal{E}_1(\lambda,\alpha,n,|z|)\geqq\mathcal{E}_1(\lambda,1,n,|z|)\\ =\{\Gamma(n+2)\Gamma(1-\lambda)-\Gamma(n+1-\lambda)\}\\ -\{(n-\lambda)\Gamma(n+1)\Gamma(1-\lambda)-\Gamma(n+1-\lambda)\}|z|.$$

Since $\mathcal{E}_1(\lambda,1,n,|z|)$ is a decreasing function of $|z|$, we also have

$$(3.23)\qquad \mathcal{E}_1(\lambda,\alpha,n,|z|)\geqq\mathcal{E}_1(\lambda,1,n,1)\\ =(1+\lambda)\Gamma(n+1)\Gamma(1-\lambda)\\ \geqq 0$$

for $-1\leqq\lambda<1$, and $n\geqq 2$.

In order to establish the second half of Theorem 7, we note that

$$(3.24)\qquad |f'(z)|\geqq 1-\max_{n\geqq 2}\frac{n(1-\alpha)}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}|z|^{n-1}$$

and

$$(3.25)\qquad |f'(z)|\leqq 1+\max_{n\geqq 2}\frac{n(1-\alpha)}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}|z|^{n-1}.$$

Define

$$(3.26)\qquad F(\lambda,\alpha,n,|z|)=\frac{n\Gamma(n-\lambda)|z|^{n-1}}{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)}$$

and

$$(3.27)\qquad F_1(\lambda,\alpha,n,|z|)=\Gamma(n+2)\Gamma(1-\lambda)\{n-(n-\lambda)|z|\}\\ -\alpha\Gamma(n+1-\lambda)\{n-(n-\lambda)|z|\}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. Since $F_1(\lambda, \alpha, n, |z|)$ is a decreasing function of α,

$$(3.28) \quad \begin{aligned} F_1(\lambda, \alpha, n, |z|) &\geqq F_1(\lambda, 1, n, |z|) \\ &= n\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\} \\ &\quad - (n-\lambda)\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\}|z|. \end{aligned}$$

Further, since $F_1(\lambda, 1, n, |z|)$ is a decreasing function of $|z|$, we get

$$(3.29) \quad \begin{aligned} F_1(\lambda, \alpha, n, |z|) &\geqq F_1(\lambda, 1, n, 1) \\ &= \lambda\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\} \\ &\geqq 0 \end{aligned}$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$. Thus we have (3.15) and (3.16).

Finally, by taking the function

$$(3.30) \quad f(z) = z - \frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)} z^2,$$

we can show that all bounds given by Theorem 7 are sharp.

COROLLARY 3. *Let the function $f(z)$ defined by (1.9) be in the class $\mathcal{T}(\alpha, \lambda)$ with $-1 \leqq \lambda < 1$ and $0 \leqq \alpha < 1$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < (1+\lambda)/\{2-\alpha(1-\lambda)\}$.*

THEOREM 8. *Let the function $f(z)$ defined by (1.9) be in the class $\mathcal{C}(\alpha, \lambda)$. Then, if $-1 \leqq \lambda < 1$,*

$$(3.31) \quad |f(z)| \geqq |z| - \frac{(1-\alpha)(1-\lambda)^2}{2\{2-\alpha(1-\lambda)\}} |z|^2$$

and

$$(3.32) \quad |f(z)| \leqq |z| + \frac{(1-\alpha)(1-\lambda)^2}{2\{2-\alpha(1-\lambda)\}} |z|^2$$

for $z \in \mathcal{U}$. Furthermore, if $0 \leqq \lambda < 1$,

$$(3.33) \quad |f'(z)| \geqq 1 - \frac{(1-\alpha)(1-\lambda)^2}{2-\alpha(1-\lambda)} |z|$$

and

$$(3.34) \quad |f'(z)| \leqq 1 + \frac{(1-\alpha)(1-\lambda)^2}{2-\alpha(1-\lambda)} |z|$$

for $z \in \mathcal{U}$. The bounds (3.31) to (3.34) are sharp.

Proof. By means of Theorem 6, we have

$$(3.35)\quad |f(z)|\geqq |z|-\max_{n\geqq 2}\left\{\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}}|z|^n\right\}$$

and

$$(3.36)\quad |f(z)|\leqq |z|+\max_{n\geqq 2}\left\{\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}}|z|^n\right\}.$$

Let

$$(3.37)\quad H(\lambda,\alpha,n,|z|)=\frac{\Gamma(n-\lambda)^2|z|^n}{\Gamma(n+1)\Gamma(1-\lambda)\{\Gamma(n+1)\Gamma(1-\lambda)-\Gamma(n-\lambda)\alpha\}}$$

and

$$(3.38)\quad H_1(\lambda,\alpha,n,|z|)=\Gamma(n+1)\Gamma(1-\lambda)\{(n+1)^2-(n-\lambda)^2|z|\}-\alpha(n-\lambda)\Gamma(n-\lambda)\{(n+1)-(n-\lambda)|z|\}$$

for $-1\leqq\lambda<1$, $0\leqq\alpha<1$, $n\geqq 2$, and $z\in\mathcal{U}$. Then, we know that $H(\lambda,\alpha,n,|z|)$ is a decreasing function of n if

$$H_1(\lambda,\alpha,n,|z|)\geqq 0.$$

In fact, since $H_1(\lambda,\alpha,n,|z|)$ is a decreasing function of α, and $H_1(\lambda, 1,n,|z|)$ is a decreasing function of $|z|$, we can prove that

$$\begin{aligned}(3.39)\quad H_1(\lambda,\alpha,n,|z|)&\geqq H_1(\lambda,1,n,|z|)\\&\geqq H_1(\lambda,1,n,|1|)\\&=(1+\lambda)\{(2n+1-\lambda)\Gamma(n+1)\Gamma(1-\lambda)\\&\quad-\Gamma(n+1-\lambda)\}\\&\geqq 0\end{aligned}$$

for $-1\leqq\lambda<1$ and $n\geqq 2$. Consequently, we have the first half of Theorem 8.

Next, we note that

$$(3.40)\quad |f'(z)|\geqq 1-\max_{n\geqq 2}\frac{1-\alpha}{\dfrac{\Gamma(n)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}}|z|^{n-1}$$

and

$$(3.41)\quad |f'(z)| \leqq 1+\max_{n\geqq 2} \frac{1-\alpha}{\frac{\Gamma(n)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}} |z|^{n-1}.$$

Define

$$(3.42)\quad G(\lambda, \alpha, n, |z|) = \frac{\Gamma(n-\lambda)^2 |z|^{n-1}}{\Gamma(n)\Gamma(1-\lambda)\{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)\}}$$

and

$$(3.43)\quad \begin{aligned} G_1(\lambda, \alpha, n, |z|) &= \Gamma(n+1)\Gamma(1-\lambda)\{n(n+1)-(n-\lambda)^2|z|\} \\ &\quad -\alpha\Gamma(n+1-\lambda)\{n-(n-\lambda)|z|\} \end{aligned}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. Then it is sufficient to prove that

$$G_1(\lambda, \alpha, n, |z|) \geqq 0.$$

Note that $G_1(\lambda, \alpha, n, |z|)$ is a decreasing function of α, and $G_1(\lambda, 1, n, |z|)$ is a decreasing function of $|z|$. Thus we have,

$$(3.44)\quad \begin{aligned} G_1(\lambda, \alpha, n, |z|) &\geqq G_1(\lambda, 1, n, |z|) \\ &\geqq G_1(\lambda, 1, n, 1) \\ &= (n+2\lambda n-\lambda^2)\Gamma(n+1)\Gamma(1-\lambda)-\lambda\Gamma(n+1-\lambda) \\ &\geqq 0 \end{aligned}$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$, which implies the second half of Theorem 8.

Finally, all bounds asserted by Theorem 8 are sharp for the function

$$(3.45)\quad f(z) = z - \frac{(1-\alpha)(1-\lambda)^2}{2\{2-\alpha(1-\lambda)\}} z^2.$$

COROLLARY 4. *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $\mathcal{C}(\alpha, \lambda)$ *with* $-1 \leqq \lambda < 1$ *and* $0 \leqq \alpha < 1$. *Then the unit disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk* $|w| < r_0$, *where* r_0 *is given by*

$$(3.46)\quad r_0 = 1 - \frac{(1-\alpha)(1-\lambda)^2}{2\{2-\alpha(1-\lambda)\}}.$$

4. Starlikeness and Convexity

Silverman [14] proved the following lemmas.

LEMMA 3. *Let the function* $f(z)$ *be defined by* (1.9). *Then* $f(z)$ *is starlike of order* α *if and only if*

$$(4.1)\quad \sum_{n=2}^{\infty} (n-\alpha) a_n \leqq 1-\alpha$$

for $0 \leqq \alpha < 1$.

LEMMA 4. *Let the function* $f(z)$ *be defined by* (1.9). *Then* $f(z)$ *is convex of order* α *if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leqq 1-\alpha \tag{4.2}$$

for $0 \leqq \alpha < 1$.

By applying the above lemmas, we now prove

THEOREM 9. *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $\mathscr{T}^*(\alpha, \lambda)$ *with* $0 \leqq \lambda < 1$ *and* $0 \leqq \alpha < 1$. *Then* $f(z)$ *is starlike of order* α.

Proof. Note that

$$n \leqq \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \tag{4.3}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. This shows that

$$\begin{aligned} \sum_{n=2}^{\infty} (n-\alpha)a_n &\leqq \sum_{n=2}^{\infty} \left\{ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha \right\} a_n \\ &\leqq 1-\alpha, \end{aligned} \tag{4.4}$$

and we complete the proof of Theorem 9 in view of (4.1).

Similarly, by using Lemma 4, we have

THEOREM 10. *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $\mathscr{C}(\alpha, \lambda)$ *with* $0 \leqq \lambda < 1$ *and* $0 \leqq \alpha < 1$. *Then* $f(z)$ *is convex of order* α.

REMARK 3. For $\lambda = 0$, the classes $\mathscr{T}^*(\alpha, \lambda)$ and $\mathscr{C}(\alpha, \lambda)$ reduce to the classes $\mathscr{T}^*(\alpha)$ and $\mathscr{C}(\alpha)$, respectively, which were introduced by Silverman [14]. It follows that

$$\mathscr{T}^*(\alpha, 0) = \mathscr{T}^*(\alpha) \tag{4.5}$$

and

$$\mathscr{C}(\alpha, 0) = \mathscr{C}(\alpha). \tag{4.6}$$

Hence, by means of Theorem 9 and Theorem 10, we have

$$\mathscr{T}^*(\alpha, \lambda) \subset \mathscr{T}^*(\alpha, 0) \qquad (0 \leqq \lambda < 1) \tag{4.7}$$

and

$$\mathscr{C}(\alpha, \lambda) \subset \mathscr{C}(\alpha, 0) \qquad (0 \leqq \lambda < 1). \tag{4.8}$$

Since

$$\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\leqq n-\alpha \tag{4.9}$$

and

$$\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\leqq n(n-\alpha) \tag{4.10}$$

for $\lambda<0$, $0\leqq\alpha<1$, and $n\geqq 2$, we also have

$$\mathscr{T}^*(\alpha,\lambda)\supset\mathscr{T}^*(\alpha,0) \qquad (\lambda<0) \tag{4.11}$$

and

$$\mathscr{C}(\alpha,\lambda)\supset\mathscr{C}(\alpha,0) \qquad (\lambda<0). \tag{4.12}$$

THEOREM 11. *Let the function $f(z)$ defined by (1.9) be in the class $\mathscr{T}^*(\alpha,\lambda)$ with $\lambda<1$ and $0\leqq\alpha<1$. Then $f(z)$ is convex of order δ $(0\leqq\delta<1)$ in the disk $|z|<r_1$, where*

$$r_1=\inf_{n\geqq 2}\left[\frac{(1-\delta)\{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)\}}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda)}\right]^{1/(n-1)}. \tag{4.13}$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|<1-\delta \tag{4.14}$$

for $|z|<r_1$. Note that

$$\left|\frac{zf''(z)}{f'(z)}\right|\leqq\frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}na_n|z|^{n-1}}\leqq 1-\delta, \tag{4.15}$$

if and only if

$$\sum_{n=2}^{\infty}n(n-\delta)a_n|z|^{n-1}\leqq 1-\delta. \tag{4.16}$$

With the aid of Theorem 3, we need only find the values of $|z|$ for which

$$|z|^{n-1}\leqq\frac{(1-\delta)\{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)\}}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda)} \qquad (n\geqq 2) \tag{4.17}$$

and the proof of Theorem 11 is thus completed.

5. Classes of Functions with Fixed Finitely Many Coefficients

In view of Theorem 3, we denote by $\mathscr{T}^*(\alpha, \lambda, k)$ the subclass of $\mathscr{T}^*(\alpha,\lambda)$ consisting of all functions of the form

(5.1) $$f(z)=z-\sum_{i=2}^{k} A_i p_i z^i-\sum_{n=k+1}^{\infty} a_n z^n,$$

where

(5.2) $$A_i=(1-\alpha)\Big/\left\{\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)}-\alpha\right\},$$

$0\leqq p_i\leqq 1$, and $0\leqq \sum_{i=2}^{k} p_i\leqq 1$.

Further, in view of Theorem 4, we denote by $\mathscr{C}(\alpha,\lambda,k)$ the sublcass of $\mathscr{C}(\alpha,\lambda)$ consisting of all functions of the form

(5.3) $$f(z)=z-\sum_{i=2}^{k} B_i p_i z^i-\sum_{n=k+1}^{\infty} a_n z^n,$$

where

(5.4) $$B_i=(1-\alpha)\Big/\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)}\left\{\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)}-\alpha\right\},$$

$0\leqq p_i\leqq 1$, and $0\leqq \sum_{i=2}^{k} p_i\leqq 1$.

Our first result on the classes of functions with fixed finitely many coefficients is contained in

THEOREM 12. *Let the function $f(z)$ be defined by* (5.1). *Then $f(z)$ is in the class $\mathscr{T}^*(\alpha,\lambda,k)$ if and only if*

(5.5) $$\sum_{n=k+1}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]$$

for $\lambda<1$ and $0\leqq\alpha<1$. The result (5.5) *is sharp.*

Proof. Putting

(5.6) $$a_i=A_i p_i \qquad (i=2,3,\dots,k)$$

in Theorem 3, we have

(5.7) $$(1-\alpha)\sum_{i=2}^{k} p_i+\sum_{n=k+1}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq 1-\alpha$$

which implies (5.5). Furthermore, the result is sharp for the functions

(5.8) $$f(z)=z-\sum_{i=2}^{k} A_i p_i z^i -\left[(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n \ (n\geqq k+1).$$

COROLLARY 5. *Let the function $f(z)$ defined by* (5.1) *be in the class $\mathscr{T}^*(\alpha,\lambda,k)$. Then*

$$(5.9)\quad a_n \leqq (1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right] \Big/ \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\} \quad (n \geqq k+1).$$

The result (5.9) *is sharp for functions* $f(z)$ *of the form* (5.8).

Similarly, we prove

THEOREM 13. *Let the function* $f(z)$ *be defined by* (5.3). *Then* $f(z)$ *is in the class* $\mathscr{C}(\alpha, \lambda, k)$ *if and only if*

$$(5.10)\quad \sum_{n=k+1}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\} a_n \leqq (1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]$$

for $\lambda<1$ *and* $0\leqq\alpha<1$. *The result* (5.10) *is sharp for the functions*

$$(5.11)\quad f(z)=z-\sum_{i=2}^{k} B_i p_i z^i - \left[(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right] \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right] z^n \quad (n\geqq k+1).$$

COROLLARY 6. *Let the function* $f(z)$ *defined by* (5.3) *be in the class* $\mathscr{C}(\alpha, \lambda, k)$. *Then*

$$(5.12)\quad a_n \leqq (1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right] \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\} \quad (n\geqq k+1).$$

The result (5.12) *is sharp for functions* $f(z)$ *of the form* (5.11).

We find the extreme points of $\mathscr{T}^*(\alpha, \lambda, k)$ and $\mathscr{C}(\alpha, \lambda, k)$ by using the same technique as in Theorem 5.

THEOREM 14. *The extreme points of* $\mathscr{T}^*(\alpha, \lambda, k)$ *are*

$$(5.13)\quad f_k(z)=z-\sum_{i=2}^{k} A_i p_i z^i$$

and

$$(5.14)\quad f_n(z)=z-\sum_{i=2}^{k} A_i p_i z^i - \left[(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right] \Big/ \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right] z^n \quad (n\geqq k+1),$$

where A_i *is given by* (5.2)

THEOREM 15. *The extreme points* $\mathscr{C}(\alpha, \lambda, k)$ *are*

$$f_k(z)=z-\sum_{i=2}^{k} B_i p_i z^i \tag{5.15}$$

and

$$f_n(z)=z-\sum_{i=2}^{k} B_i p_i z^i - \left[(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right] \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right] z^n \qquad (n \geqq k+1), \tag{5.16}$$

where B_i *is given by* (5.4).

Next we determine the radius of convexity for functions belonging to the class $\mathscr{T}^*$ (α, λ, k). Our result is given by

THEOREM 16. *Let the function* $f(z)$ *defined by* (5.1) *be in the class* $\mathscr{T}^*$ (α, λ, k). *Then* $f(z)$ *is convex in the disk* $|z|<r_2$, *where* r_2 *is the largest value for which*

$$\sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \frac{n^2(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda)-\alpha\Gamma(n-\lambda)} r^{n-1} \leqq 1 \tag{5.17}$$

for $n \geqq k+1$. *The result* (5.17) *is sharp.*

Proof. It suffices to prove that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{\sum_{i=2}^{k} i(i-1)A_i p_i r^{i-1} + \sum_{n=k+1}^{\infty} n(n-1)a_n r^{n-1}}{1-\sum_{i=2}^{k} iA_i p_i r^{i-1} - \sum_{n=k+1}^{\infty} na_n r^{n-1}} \leqq 1 \tag{5.18}$$

for $|z| \leqq r$. Note that (5.18) is true if and only if

$$\sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \sum_{n=k+1}^{\infty} n^2 a_n r^{n-1} \leqq 1. \tag{5.19}$$

By virtue of Theorem 12, we can write

$$a_n = \frac{(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha} \lambda_n \qquad (n \geqq k+1), \tag{5.20}$$

where $\lambda_n \geqq 0$ $(n \geqq k+1)$ and

$$\sum_{n=k+1}^{\infty} \lambda_n \leqq 1. \tag{5.21}$$

For each fixed r, we choose the integer $n_0=n_0(r)$ for which

$$n^2 r^{n-1} \Big/ \left\{ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha \right\}$$

is a maximum. Then we have

$$\sum_{n=k+1}^{\infty} n^2 a_n r^{n-1} \leqq \frac{n_0^2(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)} - \alpha} r^{n_0-1}. \tag{5.22}$$

Hence the function $f(z)$ is convex in $|z| \leqq r_2$ if

$$\sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \frac{n_0^2(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)} - \alpha} r^{n_0-1} \leqq 1. \tag{5.23}$$

We find the value r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{k} i^2 A_i p_i r_0^{i-1} + \frac{n_0^2(1-\alpha)\left[1-\sum_{i=2}^{k} p_i\right]}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)} - \alpha} r^{n_0-1} = 1. \tag{5.24}$$

Then this value r_0 is the radius of convexity of functions $f(z)$ in the class $\mathcal{C}^*(\alpha, \lambda, k)$. Furthermore, we know that the result of the theorem is sharp for the functions $f(z)$ given by (5.8).

6. Quasiconformal Extension

Let f be a sense-preserving homeomorphism of a domain $\mathcal{D}$. Then f is said to be a $\left[\frac{1+k}{1-k}\right]$-quasiconformal mapping of $\mathcal{D}$ if it is absolutely continuous on almost all lines parallel to the coordinate axes and

$$\left| \frac{\partial}{\partial \bar{z}} f(z) \right| \leqq k \left| \frac{\partial}{\partial z} f(z) \right| \qquad (z \in \mathcal{D}) \tag{6.1}$$

for $0 \leqq k < 1$. We say that a function $f(z)$ is in the class $\mathcal{S}_k$ if it is in the class $\mathcal{S}$ and has a $\left[\frac{1+k}{1-k}\right]$-quasiconformal extension defined on the extended complex plane (cf. [5]).

Let $\mathcal{S}^*(\alpha, \lambda)$ denote the subclass of $\mathcal{S}$ whose members satisfy the coefficient inequality (2.1) for $\lambda < 1$ and $0 \leqq \alpha < 1$. Then it follows that

(6.2) $$\mathcal{T}^*(\alpha, \lambda) \subset \tilde{\mathcal{S}}^*(\alpha, \lambda) \subset \mathcal{S}^*(\alpha, \lambda).$$

Also let $\tilde{\mathcal{K}}(\alpha, \lambda)$ denote the subclass of $\mathcal{S}$ whose members satisfy the coefficient inequality (2.7) for $\lambda < 1$ and $0 \leqq \alpha < 1$. Then we have

(6.3) $$\mathcal{C}(\alpha, \lambda) \subset \tilde{\mathcal{K}}(\alpha, \lambda) \subset \mathcal{K}(\alpha, \lambda).$$

We shall apply the following lemma due to Fait, Krzyz and Zygmunt [3] in order to prove Theorem 17 below.

LEMMA 5. *If the function $f(z)$ defined by* (1.1) *is in the class $\mathcal{A}$ and satisfies*

(6.4) $$\sum_{n=2}^{\infty} n|a_n| \leqq k,$$

then $f(z) \in \mathcal{S}_k$.

THEOREM 17. *Let the function $f(z)$ defined by* (1.1) *be in the class $\tilde{\mathcal{S}}^*(\alpha, \lambda)$ with $0 < \lambda < 1$ and $0 \leqq \alpha < 1$. Then $f(z) \in \mathcal{S}_k$, where*

(6.5) $$k = \frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}.$$

Proof. In view of (2.1) and Lemma 5, we need to prove that (2.1) implies (6.4). Define $L(\alpha, \lambda, n)$ by

(6.6) $$L(\alpha, \lambda, n) = \frac{1}{1-\alpha}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} - \frac{n}{k}$$

for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Then it suffices to show that $L(\alpha, \lambda, n) \geqq 0$ for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Note that $L(\alpha, \lambda, n) \geqq 0$ if

(6.7) $$k \geqq \frac{n(1-\alpha)\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\Gamma(n-\lambda)}$$

for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Since

$$\frac{n(1-\alpha)\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\Gamma(n-\lambda)}$$

is a decreasing function of n, and

(6.8) $$0 \leqq \frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)} < 1$$

for $0 < \lambda < 1$ and $0 \leqq \alpha < 1$, we conclude that $f(z) \in \mathcal{S}_k$ for k given by (6.5).

COROLLARY 7. *Let the function $f(z)$ defined by* (1.9) *be in the class $\mathcal{T}^*(\alpha, \lambda)$ with $0 < \lambda < 1$ and $0 \leqq \alpha < 1$. Then $f(z) \in \mathcal{S}_k$ for k given by* (6.5).

Recently, Brown [1] established the following results.

LEMMA 6. *If the function $f(z)$ defined by* (1.1) *is in the class* $\mathcal{A}$ *and satisfies*

$$\sum_{n=2}^{\infty}\left[n-\frac{1-k}{1+k}\right]|a_n|\leqq\frac{2k}{1+k}, \tag{6.9}$$

then $f(z)\in\mathcal{S}^*\cap\mathcal{S}_k$.

LEMMA 7. *If the function $f(z)$ defined by* (1.1) *is in the class* $\mathcal{A}$ *and satisfies*

$$\sum_{n=2}^{\infty}n\left[n-\frac{1-k}{1+k}\right]|a_n|\leqq\frac{2k}{1+k}, \tag{6.10}$$

then $f(z)\in\mathcal{K}\cap\mathcal{S}_k$.

THEOREM 18. *Let the function $f(z)$ defined by* (1.1) *be in the class* $\tilde{\mathcal{S}}^*(\alpha,\lambda)$ *with* $0\leqq\lambda<1$ *and* $0<\alpha<1$. *Then* $f(z)\in\mathcal{S}^*(\alpha)\cap\mathcal{S}_k$, *where* $k=(1-\alpha)/(1+\alpha)$.

Proof. Since

$$n\leqq\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \tag{6.11}$$

for $0\leqq\lambda<1$ and $n\geqq2$, we have

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}|a_n| \tag{6.12}$$
$$\leqq 1-\alpha,$$

which implies that $f(z)\in\mathcal{S}^*(\alpha)$, by means of [14].

Further, replacing α by $(1-k)/(1+k)$ in (6.12), we get (6.9). Hence $f(z)$ is also in the class $\mathcal{S}^*\cap\mathcal{S}_k$ for $k=(1-\alpha)/(1+\alpha)$. We note that $\mathcal{S}^*(\alpha)\subset\mathcal{S}^*$ for $0\leqq\alpha<1$. Therefore, $f(z)\in\mathcal{S}^*(\alpha)\cap\mathcal{S}_k$.

COROLLARY 8. *Let the function $f(z)$ defined by* (1.9) *be in the class* $\mathcal{T}^*(\alpha,\lambda)$ *with* $0\leqq\lambda<1$ *and* $0<\alpha<1$. *Then* $f(z)\in\mathcal{T}^*(\alpha,0)\cap\mathcal{S}_k$, *where* $k=(1-\alpha)/(1+\alpha)$.

By using Lemma 7, we prove

THEOREM 19. *Let the function $f(z)$ defined by* (1.1) *be in the class* $\tilde{\mathcal{K}}(\alpha,\lambda)$ *with* $0\leqq\lambda<1$ *and* $0<\alpha<1$. *Then* $f(z)\in\mathcal{K}(\alpha)\cap\mathcal{S}_k$, *where* $k=(1-\alpha)/(1+\alpha)$.

COROLLARY 9. *Let the function $f(z)$ defined by* (1.9) *be in the class*

$\mathcal{C}(\alpha, \lambda)$ *with* $0 \leqq \lambda < 1$ *and* $0 < \alpha < 1$. *Then* $f(z) \in \mathcal{C}(\alpha, 0) \cap \mathcal{S}_k$, *where* $k = (1-\alpha)/(1+\alpha)$.

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