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THE CLOSED IDEALS OF AN INFINITE SEMIFINITE FACTOR

SA GE LEE, SUNG JE CHO AND SUNG KI KIM

1. Introduction.

Throughout the paper, H denotes an infinite dimensional complex Hilbert sapce, $B(H)$ the set of all (bounded, linear) operators on H. Even if it has been a folklore in operator theory to characterize the nonzero closed (two sided) ideals in $B(H)$ ([6], [9], [13]) and the ideal of relatively compact operators in a II_∞-factor ([1], [2], [7], [8], [14]), there has'nt appeared a unified theory for this characterizing problem, to the best of my knowledge, governing both the cases of I_∞-factors and II_∞-factors.

The purpose of the paper is to show that there is such a unified way for characterization of the nonzero closed ideals in an infinite semifinite factor, based on our earlier work [11].

2. Ideals.

LEMMA 1. *Any nonzero left ideal I_* of a von Neumann algebra A contains a nonzero projection.*

Proof. Let $x \in I_*$ and let $x = u|x|$ be the left polar decomposition of x. Since the partial isometry $u \in A$, we see that $|x| = u^* x \in I_*$. We may assume that $|x| \neq 0$ without loss of generality. There is $\varepsilon > 0$ such that $E[\varepsilon, \infty) \neq 0$, where $E(\cdot)$ is the spectral measure of $|x|$. We abbreviate $E[\varepsilon, \infty)$ to e. Then range $(|x|e) \subset$ range $(|x|)$. Since the origin of the complex plane is an isolated point of the spectrum $\sigma(|x| e)$, we see that range$(|x|e)$ is closed. Let p be the nonzero range projection of $|x|e$. Thus $p = |x|y$ for some $y \in A$, by the strengthening ([11] Lemma 1.) of R. G. Douglas result ([4] p. 413 Theorem 1),

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since range$(p) \subset$ range$(|x|)$. It follows that $p = y^*|x| \in I_*$.

LEMMA 2. *If I_* is an ideal in a factor A and contains an infinite projection p (relative to A), then I_* contains every finite projections of A.*

Proof. Let q be an arbirary finite projection of A. Since A is a factor and p is infinite, we see that $q \prec p$. Hence $q \sim p_1 \leq p$. Since $p_1 = p_1 p \in I_*$, we can easily show that $q \in I_*$.

LEMMA 3. *Let I_* be an ideal in a factor A and contain a nonzero finite projection p of A. Then I_* contains every finite projection q of A.*

Proof. Let p be a nonzero finite projection of A such that $p \in I_*$, and let q be any finite projection of A. If $q \prec p$, then a similar argument as above Lemma 2, we see that $q \in I_*$. Now, let $p \prec q$. Let $\{q_i\}_{i \in \Gamma}$ be a maximal family of mutually orthogonal subprojections of q such that $p \sim q_i$, for every $i \in \Gamma$. Since $\sum_{i \in \Gamma} q_i \leq q$ and q is finite, we see that the cardinality card(Γ) of Γ must be finite. Consequently $\sum_{i \in \Gamma} q_i \in I_*$. Now, $p \not\prec q - \sum_{i \in \Gamma} q_i$ by the maximality of $\{q_i\}_{i \in \Gamma}$. Consequently, $q - \sum_{i \in \Gamma} q_i \prec p$. Again, this implies that $q - \sum_{i \in \Gamma} q_i \in I_*$. Thus, $q = \sum_{i \in \Gamma} q_i + (q - \sum_{i \in \Gamma} q_i) \in I_*$.

THEOREM 1. *If I_* is a nonzero ideal in a factor A, then I_* contains all the finite projections of A.*

Proof. Combine the above lemmas 1, 2 and 3.

DEFINITION 1. Let e be a projection in an infinite semifinite von Neumann algebra A. When e is a finite projection relative to A, we say that e has the *relative finite rank* and denote this fact by the symbol rank$(e) < \aleph_0$. Now let e be an infinite projection of A and assume further that there exists a family $\{e_i\}_{i \in \Gamma}$ of mutually orthogonal nonzero finite projections in A such that $e = \sum_{i \in \Gamma} e_i$. Then we define the *relative rank* rank(e) of e by rank(e) = card(Γ), where card(Γ) is the cardinality of Γ ([11] Definition 1).

By (p. 252 [3] Lemma 6), it is clear that rank(e) of an infinite projection e in A is well defined independent of the choice of the expressions $e=\sum_{i\in\Gamma} e_i$ and that rank$(e)\leq$rank(f), whenever $e\prec f, e, f$ are infinite projections in A.

For $x\in A$, let $l(x)$ denote the left support of x in A that is the range projection of x in A. For every infinite cardinal α, we define $I_\alpha=\{x\in A : \text{rank } (l(x))<\alpha\}$ and $J_\alpha=\bar{I}_\alpha$, the norm closure of I_α in A ([11] Definition 3).

For an infinite cardinal α the *relative α-topology* T_α on H is defined as the locally convex topology on H, generated by the set of all seminorms p_M of the form

$$x\in H\rightarrow\sup\{|(x, y) : y\in M_1\},$$

where M varies over F_α and F_α is the set of all nonzero closed subspaces of H each of whose range projections belongs to A with the relative rank$<\alpha$. Here M_1 denotes the unit ball of M ([11] Definition 4).

We put S as the set of all T_α-neighborhoods s of 0 in H such that s is a finite intersection of sets $\{y\in H : p_M(y)<\varepsilon\}$, where M varies over F_α and ε varies over the set of all positive numbers. We regard S as a directed set with respect to the order $s\leq t$, meaning $t\subset s$.

The followingis obtained in ([11] Theorem 1).

THEOREM A. *Let A be an infinite semifinite von Neumann algebra, P the set of all projections of A, α an infinite cardinal. The next six statements are pairwise equivalent.*

(i) $x\in J_\alpha$.

(ii) *If $q\in P$, $q(H)\subset x(H)$, then rank $(q)<\alpha$.*

(iii) *If $p\in P$ and x is bounded below on $P(H)$, then rank$(p)<\alpha$.*

(iv) *For every $\varepsilon>0$, there exists $p\in P$ such that $\|xp\|\geq\varepsilon$ and rank $(I-p)<\alpha$.*

(v) *$x|H_1 : H_1\rightarrow H$ is norm continuous with respect to T_α on H_1 and the norm on H, where H_1 denotes the unit ball of H.*

(vi) *For every norm bounded net $\{\xi_s : s\in S\}$ in H such that $\xi_s\rightarrow 0$ in T_α, we have $x\xi_s\rightarrow 0$ in norm.*

It is the main purpose of our paper to show that these J_α's just constitute the set of all nonzero closed ideals of A when A is an infinite semifinite factor (See Theorem 5). From now on, we only consider the case when A is an infinite semifinite factor.

LEMMA 4. *Let e be an arbitrarily given nonzero finite projection in A. Then there exists an infinite family $\{e_i\}_{i\in\Gamma}$ of mutually orthogonal* *nonzero finite projections in* A *such that* $p=\sum_{i\in\Gamma} e_i$ *and that* $e_i\sim e$ *for all* $i\in\Gamma$, where p is any given infinite projection in A.

Proof. By comparability of two projections in a factor and Zorn's lemma, one can find a maxmal family $\{p_i\}_{i\in\Gamma}$ of nonzero mutually orthogonal finite subprojections of p in A such that $p_i\sim e$ for all $i\in\Gamma$. We write $f=p-\sum_{i\in\Gamma} p_i$. Then $f(\sum_{i\in\Gamma} p_i)=0$ and $f\prec p_i$, for all $i\in\Gamma$, by the maximality of $\{p_i\}_{i\in\Gamma}$. By the same proof of (II) in ([15] pp. 97–98), we see that there exists a family $\{e_i\}_{i\in\Gamma}$ of nonzero finite mutually orthogonal subprojections e_i of p such that $p=\sum_{i\in\Gamma} e_i$ and that $e_i\sim e$ for all $i\in\Gamma$.

DEFINITION 2. (An analogue to Definition 6. 2 in [13] p. 604) Let A be an infinite semifinite factor and let I_* be a nonzero ideal of A. The *(relative) height* $h(I_*)$ of I_* is defined by

$$h(I_*)=\sup\{rank(p) : p\in P\},$$

when I_* contains at least one infinite projection of A, and

$$h(I_*)=\aleph_0,$$

when I_* does not contain any infinite projection of A (See Theorem 1). We call that $h(I_*)$ is *accessible* if there is an infinite projection $p\in P$ in A such that $\mathrm{rank}(p)=h(I_*)$. Otherwise, the height $h(I_*)$ is called *inaccessible*.

If the height $h(I_*)$ is inaccessible, then it is immediate to see that $h(I_*)$ is a limit cardinal (p. 604 [13]).

LEMMA 5. *If* p *and* q *are projections in* A *such that* $rank(p)=rank(q)\geq\aleph_0$, *then* $p\sim q$.

Proof. We fix a nonzero finite projection e in A. By Lemma 4, there exist two families $\{e_i\}_{i\in\Gamma}$ and $\{f_i\}_{i\in\Gamma}$ each of which consists of nonzero finite mutually orthogonal projections such that $e_i\sim e\sim f_i$ for all $i\in\Gamma$ and that $p=\sum_{i\in\Gamma} e_i$, $q=\sum_{i\in\Gamma} f_i$. Consequently, $p\sim q$.

LEMMA 6. *Let* I_* *be a nonzero ideal of* A. *Let* p *be an infinite projection in* A *such that* $p\in I_*$. *Then for any projection* q *in* A *such that* $rank(q)\leq rank(p)$, *we have that* $q\in I_*$ (*cf. p.* 604 [13] *Lemma* 6. 1)

Proof By Lemma 2, we may assume that q is an infinite projection. If $q\prec p$, then by a similar argument as the proof of Lemma 2, one sees that $q\in I_*$. Let $p\prec q$, then $\mathrm{rank}(p)\leq\mathrm{rank}(q)$. Hence $\mathrm{rank}(p)=\mathrm{rank}(q)$ with the aid of the hypothesis of the present lemma. Lemma 5 then implies that $p\sim q$.

THEOREM 2. *Let I_* be a two sided ideal of A and let $\beta=h(I_*)\geq\aleph_0$. If the height $h(I_*)$ is accessible, then $I_*=J_{\beta+1}$ (cf. Theorem* 6.2 [13] *p.* 604)

Proof. Let $x\in I_*$. We want to apply (ii)→(i) of Theorem A. Assume that $p\in P$ be any projection in A satisfying $p(H)\subset x(H)$. Then $\mathrm{rank}(p)\leq h(I_*)=\beta<\beta+1$, noticing $p\in I_*$ ([11] Lemma 1). By (ii) → (i) of Theorem A, we see that $x\in J_{\beta+1}$. Conversely, let $x\in J_{\beta+1}$. Then there is a sequence: $\{x_n\}\subset I_{\beta+1}$ such that $\|x_n-x\|\to 0$. Thus, $\mathrm{range}(x)\subset\bigvee_{n=1}^{\infty}\overline{\mathrm{range}(x_n)}$, the closed linear span of range (x_n)'s. On the other hand rank $(l(x_n))<\beta+1$, that is, $\mathrm{rank}(l(x_n))\leq\beta$, for every n. It is easy to see that the relative rank of the projection whose range is $\bigvee_{n=1}^{\infty}\mathrm{range}(x_n)$ is $\leq\aleph_0\beta=\beta$. Consequently, $\mathrm{rank}(l(x))\leq\beta<\beta+1$. Since $h(I_*)$ is accessible, there is $p\in P\cap I_*$ such that $\beta=\mathrm{rank}(p)$. Since $\mathrm{rank}(l(x))\leq\beta=\mathrm{rank}(p)$, we see that $l(x)\in I_*$ by Lemma 6. Hence $x=l(x)x\in I_*$.

THEOREM 3. *Let I_* be an ideal of A. Let $\beta=h(I_*)$ be an infinite cardinal which is also inaccessible, then β is a limit cardinal number and $I_\beta\subset I_*\subset J_\beta$ (cf. Theorem* 6.3 [13] *p.* 605).

Proof. If $x\in I_\beta$, then $\mathrm{rank}(l(x))<\beta$. As we already mentioned that an inaccessible $h(I_*)$ $(=\beta)$ is a limit cardinal, there exists a cardinal β_1 such that $\mathrm{rank}(l(x))<\beta_1<\beta$. By definition of $h(I_*)$, there is $p\in P\cap I_*$ such that $\beta_1\leq\mathrm{rank}(p)$. Hence $\mathrm{rank}(l(x))<\mathrm{rank}(p)$. If β_1 is an infinite cardinal so is $\mathrm{rank}(p)$. Even if β_1 is a finite cardinal, it can be taken arbitrarily large. Hence, again $\mathrm{rank}(p)$ is an infinite cardinal. Thus, by Lemma 6, $x\in I_*$.

Now let $x\in I_*$. For any $p\in P$ with $p(H)\subset\mathrm{range}(x)$, we see that $p\in I_*$ by Lemma 1 of [11]. Then $\mathrm{rank}(p)<h(I_*)$ by the inaccessibility of $h(I_*)$. Hence $\mathrm{rank}(p)<\beta$. By (ii) → (i) of Theorem A, we see that $x\in J_\beta$.

THEOREM 4. *Let $\beta\geq\aleph_0$ be a cardinal number. Let β be not a limit cardinal number. Then $I_\beta=J_\beta$ (cf. Theorem* 6.4 [13] *p.* 605).

Proof. $h(I_\beta)\le\beta$. Note that rank$(p)\neq\beta$, for any $p\in I_\beta\cap P$. Thus rank$(p)\le\beta-1$, since β is not a limit cardinal. On the other hand one can easily show that there is $q\in P\cap I_\beta$ such that rank$(q)=\beta-1$. Conequently $h(I_\beta)=\beta-1$. By Theorem 2, $I_\beta=J_{(\beta-1)+1}=J_\beta$.

THEOREM 5. *Let J be a nonzero closed ideal in an infinite semifinite factor A. Then there is a cardinal humber α; $\aleph_0\le\alpha\le h+1$, where $h=$ rank(I) (See Lemma 4) suchhat $J=J_\alpha$ (cf. Corollary 6.2 [13] p. 605).*

Proof. Let us put $\beta=h(J)$. Then $\beta\ge\aleph_0$ by Theosrem 1 and Definition 2. When $h(J)$ is accessible, we get $J=J_{\beta+1}$ by Theorem 2. When $h(J)$ is inaccessible, $I_\beta\subset J\subset J_\beta$, by Theorem 3. Thus $J=\bar{I}_\beta=J_\beta$.

REMARK. In [12], the set of selfcommutators in an infinite semifinite factor has been characterized by introducing the notion of weighted spectra with respect to J_h, $h=$rank(I) (cf. [5]).

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Seoul National University
Seoul 151, Korea