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CHARACTERISTIC PROPERTIES OF SOLUBLE GROUPS OF FINITE RANK

A. H. RHEMTULLA*

1.0 Of the many well known and basic propeties of nilpotent groups we survey a few with the intention of looking at the finiteness conditions they impose on a finitely generated soluble group. Interestingly they all restrict the rank of the group and seem to be characteristic properties of soluble groups of finite rank. Several open questions are asked on the same theme.

1.1 **Finite index intersection (*FII*) property.** In [1] L. Greenberg proved that for finitely generated subgroups U, V of a free group, $|U:U\cap V|<\infty$ and $|V:U\cap V|<\infty$ imply $|\langle U, V\rangle : U\cap V|<\infty$. We call G an *FII-group* if $|U:U\cap V|<\infty$ and $|V:U\cap V|<\infty$ imply $|\langle U, V\rangle : U\cap V|<\infty$ for all subgroups U, V of G. Nilpotent groups have this property but the infinite dihedral group $\langle x, y\,;\, x^2=y^2=1\rangle$ does not, as can be seen by taking $U=\langle x\rangle$, $V=\langle y\rangle$. Let G be a finitely generated soluble group. If G is an *FII-group*, then it is periodic-by-nilpotent-by-free abelian, with the nilpotent section having finite rank. Conversely if G is periodic-by-nilpotent-by-free abelian with the nilpotent section having finite rank, then it has an *FII-subgroup* of finite index ([6], Corollary D).

1.2 **The isolator property.** If $H\le K\le G$, then denote by $K\sqrt{H}$ the set $\{g\in K\,;\, g^n\in H$ for some $n\ge 1\}$, the so called *isolator* of H in K. We say H is *isolated* in K if $K\sqrt{H}=H$; and G has the *isolator property* if $G\sqrt{H}$ is a subgroup for all $H\le G$. If H, K are subgroups of a group G then we write $H\sim K$ to mean $G\sqrt{H}=G\sqrt{K}$. Clearly $\sim$ is an equivalence relation on the set of all subgroups of G and, by Zorn's lemma, there is a maximal element in each equivalence

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class. If G has the isolator property then for each subgroup H of G, the equivalence class $[H]$ has the unique maximal element $G\sqrt{H}$.

Denote by (P0) the class of groups having a unique maximal element in each equivalence class $[H]$, $H \leq G$. The group in the example given at the end of section 4 of [5] is in this class but does not have the isolator property for subgroups. Interestingly, for a soluble group G the property (P0) is equivalent to the property (P2) given in section 1.3 below, and in particular for a finitely generated solvable group G, it coincides with the finite index intersection property.

QUESTION 1. For which classes of groups do properties (P0) and (P2) coincide?

It is well known, from the pioneering work of P. Hall (see [2]), that nilpotent groups have the isolator property. For a finitely generated torsionfree soluble group G, the isolator property implies finiteness of rank of G. Conversely if G has finite rank then it has a subgroup of finite index having the isolator property. A similar result holds for finitely generated linear groups ([8], Theorems B and C; [5], Theorem C).

1.3 The isolator property for normal (subnormal) subgroups. We say G has the *isolator property for normal (subnormal) subgroups* if $G\sqrt{H}$ is a subgroup whenever H is a normal (subnormal) subgroup of G. These properties are much weaker than the isolator property which is a subgroup-closed property. To get meaningful results we need to look at subgroup-closed classes with these properties. For a finitely generated soluble group the following are equivalent ([6], Theorem A).

(P1) G is an *FII-group* (see section 1.1)
(P2) for all $H \lhd K \leq G$, $K\sqrt{H}$ is a subgroup
(P3) for all H sn $K \leq G$, $K\sqrt{H}$ is a subgroup.
(H sn K denotes that H is subnormal in K).

We shall add conditions (P4), (P5) in section 1.5 and (P6), (P7) in section 1.6 to the above list.

1.4 Frattini type property. starting with the basic result that if G is soluble, H sn G and $HG' = G$ then $H = G$ it is easily established that if G is finite-by-nilpotent, $H \leq G$ and $HG' \sim G$, then $H \sim G$. In looking for the converse one needs sn-closed classed with above property. For a finitely generated soluble group G, the following are equivalent ([5], Theorem A).

(Q1) G is finite-by-nilpotent

(Q2) If $H \leq K$ sn G and $HK' \sim K$ then $H \sim K$.

Again we shall add (Q3) to this list in section 1.7. One could also add Robinson's results ([10], Theorem 10.54) on groups having the subnormal intersection property to this list.

1.5 **Weaker conditions than (Q2) above are the following:**

(P4) If $H \lhd K \leqq G$ and $HK' \sim K$ then $H \sim K$.

(P5) If H sn $K \leqq G$ and $HK' \sim K$ then $H \sim K$.

Since there exist torsion-free polycyclic groups G with G/G' finite, condition (P4) is not satisfied by polycyclic groups in general. For a finitely generated soluble group, the conditions (P4), (P5) and (P1) are equivalent ([6], Theorem B and Proposition C).

1.6 **Commutator equivalence.** A beautiful result of P. Hall ([2], Theorem 4.6) on isolators in nilpotent groups is the following. Let θ be any word in the n variables and let H_i, K_i be subgroups of a locally nilpotent group G such that $H_i \sim K_i$ for each $i=1, 2, \ldots, n$. Then $\theta(H_1, \ldots, H_n) \sim \theta(K_1, \ldots, K_n)$. If G is a finitely generated soluble group with the isolator property and H_i, K_i subgroups of G with $H_i \sim K_i$, $i=1, 2$; then by Steps 1 and 2 of the proof of Theorem C of [5] and Theorem E of [8], $[H_1, H_2] \sim [K_1, K_2]$ and $\phi(H_1) = \phi(K_1)$ where ϕ is any outer commutator word.

Question 2. If G is a finitely generated soluble group with the isolator property, H_i, K_i are subgroups of G such that $H_i \sim K_i$, $i=1, 2, \ldots, n$ and θ is a word in n variables, does it follow that $\theta(H_1, \ldots, H_n) \sim \theta(K_1, \ldots, K_n)$?

The answer is not known even for polycyclic groups having the isolator property. The commutator subgroup is of course the most important verbal subgroup and it is natural to look at finitely generated soluble groups under each of the following conditions.

(P6) If $H \lhd K \leq G$ and $H \sim K$ then $H' \sim K'$.

(P7) If H sn $K \leq G$ and $H \sim K$ then $H' \sim K'$.

(R1) If $H \leq K \leq G$ and $H \sim K$ then $H' \sim K'$.

For a finitely generated soluble group G the conditions (P6), (P7), and (P1) are equivalent ([6], Theorem B and Proposition C).

QUESTION 3. Let G be a finitely generated soluble group satisfying conditon (R1). Does G have the isolator property?

Such a group G satisfies conditions (P1) through (P7), but these are not enough to ensure the isolator property even if G is torsion-free ([5], Example in Section 4).

1.7 **The π-isolator property.** Let π be a non-empty set of primes. A natural number is called a π-number precisely if all of its prime factors are in π. A group G is said to have the *π-isolator property* if for every $H \leq G$, $\pi\sqrt{H} = \{g \in G\ ;\ g^n \in H$ for some π-number $n\}$ is a subgroup. A locally nilpotent group has the π-isolator property for any non-empty set π of primes. Where π is the set of all primes, the notion of π-isolator coincides with that of isolator. If π is any finite non-empty set of primes and G is a finitely generated soluble group with the π-isolator property, then G is finite by torsion-free nilpotent, where the finite subgroup is the direct product of π-subgroup and π'-subgroup. The converse also holds ([9] Theorem A and [5], Theorem D). In particular, for a finitely generated soluble group G, the condition (Q1) or (Q2) of section 1.4 coincides with the following: (Q3) G has the π-isolator property for some finite non-empty set π of primes.

1.8 **Finite index property.** A group G has the finite index property if H, K are subgroups of G and A, B are subgroups of finite index in H, K respectively then $|\langle H, K\rangle : \langle A, B\rangle| < \infty$. Nilpotent groups of finite rank have this property. H. Smith ([11], Theorem 1) has shown that for groups of finite rank this property holds for any pair of subgroups H and K which are subnormal in their join. It turns out that for a finitely generated torsionfree soluble group G, the finite index property coincides with the isolator property. It is easy to see that the finite index property implies the isolator property. The converse is given in [7], Theorem 2.

1.9 **The strong isolator property** We say G has the *strong isolator property* if it has the isolator property and in addition $|G\sqrt{H} : H| < \infty$ for all $H \leq G$. A finitely generated soluble group with this property is polycyclic and conversely every polycyclic group has a subgroup of finite index with this property ([8], Theorem A]. One can similarly strengthen other properties introduced in section 1.4, 1.5 and 1.6 by replacing the relation $\sim$ with $\approx$ given by: If H, K are subgroups of G, then $H \approx K$ if $|H : H \cap K| < \infty$ and $|K : H \cap K| < \infty$. Clearly $\approx$ is an equivalence relation on the set of subgroups of G and one would suspect that this stronger condition would force the group to be polycyclic.

2.0 **More questions.** There are many other reasonable conditions one could put on a finitely generated soluble group G that would result in finiteness of rank of G. The following three are a typical sample of such conditions.

2.1 **Elliptic sets and pseudo-identity automorphisms**: If A is a set and G the group generated by A, then each g in G has the form $G = a_{i_1}^{\varepsilon_1} \dots a_{i_r}^{\varepsilon_r}$, $r \geq 0$, $a_i's$ in A and $\varepsilon_i = \pm 1$. Call the least such r the A-length of g. Let A_n be the set of elements of G of A-length at most n. Then $A_n \subseteq A_{n+1}$. If for some n, $A_n = A_{n+1}$, then $A_n = G$ and in this case A is called an elliptic set of G. The terminology is due to P. Hall (Lectures at University of Cambridge, Lent term 1967). If no such n exists then A is called a parabolic set of G. If G is a finitely generated nilpotent group and H, K are subgroups of G then $H \cup K$ is an elliptic set for $\langle H, K \rangle$ This was shown by D. Robinson. The proof is simple if one uses induction on the subnormal defect of H in $\langle H, K \rangle$. From this result it follows that if G is nilpotent, generated by the set $\{a_1, \dots, a_m\}$ and if $A_i = \langle a_i \rangle$, then $\bigcup_{i=1}^{m} A_i$ is an elliptic set. Let us call a group G elliptic if $H \cup K$ is an elliptic set for $\langle H, K \rangle$ for all subgroups H, K of G. Clearly the infinite dihedral group is not elliptic.

QUESOTIN 4. Is a soluble group with the strong isolator property elliptic?

If the answer to the above question is no, then it would still be of interest to know if a polycyclic group has an elliptic subgroup of finite index.

Closely linked to the above concept is that of a pseudo-identity automorphism introducted by P. Hilton and J. Roitberg (see [4]). An automorphism ϕ of a group A is called a pseudo-identity automorphism of A if for each a in A, there is a finitely generated subgroup

$K=K(\phi, a)$ of A such that $a\in K$ and $\phi|K$ is an automorphism of K. In particular if A satisfies the maximal condition locally (i. e. A is locally max) then ϕ is a pseudo-identity automorphism of A if and only if $\langle a^{(\phi)}\rangle=\langle\phi^n a; n\in\mathbf{Z}\rangle$ is finitely generated for all a in A. If A is locally max $\phi_1, \phi_2, \ldots, \phi_k$ are pseudo-identity automorphisms of A, then for each a in A, the group $\langle\phi_k^{\gamma_k}\phi_{k-1}^{\gamma_{k-1}}\ldots\phi_1^{\gamma_1} a;\ \gamma_1 \in\mathbf{Z},\ i=1, \ldots, k\rangle$ is finitely generated. This lemma is due to Hilton. From this he concludes that if A is locally max, G is a locally nilpotent subgroup of Aut A (the group of automorphisms of A), and G is generated by pseudo-identity automorphisms of A, then every element of G is a pseudo identity automorphism of A. One can obtain this result from the above lemma of Hilton using the fact that G is an elliptic group. Once again note that if G is the infinite dihedral group generated by x, y with $x^2=y^2=1$, then we may consider G as subgroup of Aut A where A is the base group of $\langle a^G\rangle$, with $\langle a\rangle$ an infinite cyclic group. Then xy is not a pseudo-identity automorphism although x and y are.

We will say that a group G satisfies the pseudo-identity condition (pi-condition) if whenever $G\leq\mathrm{Aut}A$ where A is locally max, and G is generated by pseudo-identity automorphisms of A then every element of G is a pseudo-identity automorphism of A.

QUESTION 5. Must a finitely generated torsion-free soluble group satisfying the pi-condition have finite rank?

QUESTION 6. Does a polycyclic group have a subgroup of finite index satisfying the pi-condition?

2.2 **Abelian quotient properties**: Let G be a finitely generated soluble group. A number of properties of G can be detected by abelian quotients of certain subgroups of G. We present one illustration. If $a\in G$ then write $A=\langle a^G\rangle$. If A/A' is finitely generated for all a in G, then G is polycyclic. This is easily established using induction on the solubilty length of G and observing that the property above is quotient closed. In a similar way one can ask the following.

QUESTION 7. Let G be a finitely generated soluble group. For a in G, denote $\langle a^G\rangle$ by A and suppose that the rank of A/A' is finite for all a in G. Does it follow that G has finite rank?*

2.3 **Group whose torsion-free sections are R-groups**: Recall that an R-group is one in which $x^n=y^n$, $n>0$ implies $x=y$. If G is a soluble group then G is an R-group if and only if G is a torsion-free group having the isolator property for abelian subgroups (i. e. $G\sqrt{A}$ is a subgroup whenever A is an abelian subgroup of G). In the general case where G is not soluble one needs the extra condition that $G\sqrt{A}$ be abelian. Soluble groups having the isolator property for some other classes of subgroups would be a natural situation to look at. If G is a torsion-free soluble group of finite rank, then it has a subgroup of finite index in which every torsion-free section is an R-group ([3], Theorem F and Lemma 16). In the opposite direction a reasonable question that could be asked is the following.

QUESTION 8. Let G be a finitely generated torsion-free soluble group. If every torsion-free section of G is an R-group does it follow that G has finite rank?**

Our last question is the following.

QUESTION 9. Let G be the join of two normal subgroups H and K which are locally soluble of finite rank. Is G locally soluble of finite rank?

The question is not new, but we include it because of its importance. It is well known that the join of two locally polycyclic group is locally polycyclic. On the other hand P. Hall's example (see [10], Theorem 8.19.1) shows that the join of two normal locally soluble groups need not be locally soluble. It is therefore natural to look for a subclass of locally soluble groups that contains locally polycyclic groups and is closed under subnormal joins.

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Department of Mathematics
University of Alberta
Edmonton, Canada
T6G 2G1