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JOINS OF FINITELY BASED LATTICE VARIETIES*

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1. Introduction

The join of two finitely based lattice varieties need not be finitely based. The earliest counterexample was obtained by Baker, although it was not published until in [2]. In [5], Jónsson gave another example and also a sufficient condition for the join of two finitely based lattice varieties to be finitely based. The purpose of this paper is to establish more general results of the same nature.

2. Preliminaries

Given two quotients (intervals) p/q and r/s in a lattice, if $r=p+s$ and $s\geqslant q$, then we say that p/q *transposes weakly up* onto r/s (in symbols, $p/q\nearrow^w r/s$), and dually, if $qr=s$ and $r\leqslant p$, then we say that p/q *transposes weakly down* onto r/s (in symbols, $p/q\searrow_w r/s$). If there exists a sequence of quotients $p/q=x_0/y_0$, x_1/y_1, ..., $x_n/y_n=r/s$ in a lattice such that, for each $i<n$, x_i/y_i transposes weakly up or down onto x_{i+1}/y_{i+1}, then we say that p/q *projects weakly* onto r/s. If both $p/q\nearrow^w r/s$ and $r/s\searrow_w p/q$, that is, $p+s=r$ and $ps=q$, then we say that p/q *transposes up* onto r/s and r/s *transposes down* onto p/q (in symbols, $p/q\nearrow r/s$ and $r/s\searrow p/q$). If there exists a sequence of quotients $p/q=x_0/y_0$, x_1/y_1, ..., $x_n/y_n=r/s$ in a lattice such that, for $i<n$, x_i/y_i transposes up or down onto x_{i+1}/y_{i+1}, then we say that p/q *projects* onto r/s.

For any lattice L, if there exists a nonnegative integer r such that, for all $a,b,c,d\in L$ with $a<b$ and $c<d$, whenever b/a projects weakly onto d/c we have that b/a projects weakly onto a nontrivial subquotient of d/c in r steps, then the smallest such r is denoted by $R(L)$. But if no such r exists, then we write $R(L)=\infty$. For a class $\boldsymbol{K}$ of lattices,

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$R(\boldsymbol{K})$ denotes the supremum of $R(L)$ for $L \in \boldsymbol{K}$. (Refer to Baker [1] for a more general notion of $R(\cdot)$.)

The following four lemmas will be used to prove our main theorem in the next section. The first lemma gives the basic criterion used by Jónsson to prove his result, and it will serve a similar role here. The second lemma can also be found in [5]. The third lemma is really an old fact from universal algebra. The fourth lemma is the only one that will be supplied with a proof, but even it is essentially contained in the literature (cf. Baker [1]).

LEMMA 2.1. *Suppose $\boldsymbol{U}$ is a lattice variety, and let $\boldsymbol{V}$ and $\boldsymbol{V}'$ be subvarieties of $\boldsymbol{U}$ defined, relative to $\boldsymbol{U}$, by the identities $\alpha=\beta$ and $\gamma=\delta$, respectively, where the inclusions $\alpha \leqslant \beta$ and $\gamma \leqslant \delta$ hold in $\boldsymbol{U}$. In order for $\boldsymbol{V}+\boldsymbol{V}'$ to be finitely based relative to $\boldsymbol{U}$, it is necessary and sufficient that there exists a positive integer n with the following property:*
$P(n)$: For any $L \in \boldsymbol{U}$, if there exist $u, u' \in {}^{\omega}L$ and $c, d \in L$ with $c<d$ such that both $\beta(u)/\alpha(u)$ and $\delta(u')/\gamma(u')$ project weakly onto d/c, then there exist $v, v' \in {}^{\omega}L$ and $c', d' \in L$ with $c'<d'$ such that both $\beta(v)/\alpha(v)$ and $\delta(v')/\gamma(v')$ project weakly onto d'/c' in n steps.

For a lattice variety $\boldsymbol{V}$ and a nonnegative integer n, let $F_{\boldsymbol{V}}(n)$ be the n-generated $\boldsymbol{V}$-free lattice and $(\boldsymbol{V})^n$ the variety defined by all identities in n variables that hold in $\boldsymbol{V}$.

LEMMA 2.2. *If $F_{\boldsymbol{V}}(n)$ and $F_{\boldsymbol{V}'}(n)$ are finite for lattice varieties $\boldsymbol{V}$ and $\boldsymbol{V}'$ and for a nonnegative integer n, then $(\boldsymbol{V}+\boldsymbol{V}')^n$ is finitely based.*

LEMMA 2.3. *For any lattice L and any prime quotient q/p in L, there exists a unique subdirectly irreducible quotient lattice $\bar{L}$ of L in which $\bar{q}/\bar{p}$ is a critical quotient, where $\bar{x}$ denotes the element in $\bar{L}$ corresponding $x \in L$.*

LEMMA 2.4. *Let $\bar{L}$ be a quotient lattice of a lattice L and let q/p be a prime quotient in L. For all $a, b \in L$ with $a<b$ and a nonnegative integer n, if $\bar{b}/\bar{a}$ projects weakly onto $\bar{q}/\bar{p}$ in n steps, then b/a projects weakly onto q/p in $(n+1)$ steps when $n>0$, and in two steps when $n=0$.*

Proof. Suppose that $\bar{b}/\bar{a}$ projects weakly onto $\bar{q}/\bar{p}$ in 0 step, i.e., $\bar{a}=\bar{p}$ and $\bar{b}=\bar{q}$. Then

$$b/a \nearrow_w (b+p)/(a+p) \searrow_w (b+p)q/(a+p)q = q/p,$$
$$b/a \searrow_w bq/aq \nearrow_w (bq+p)/(aq+p) = q/p,$$

since $p \leqslant (a+p)q < (b+p)q \leqslant q$ and $p \leqslant aq+p < bq+p \leqslant q$.

Now suppose that $\bar{b}/\bar{a}$ projects weakly onto $\bar{q}/\bar{p}$ in n steps with $n > 0$. Since the other cases can be treated similarly, we may assume

$$\bar{b}/\bar{a} \nearrow_w \bar{b}_1/\bar{a}_1 \searrow_w \bar{b}_2/\bar{a}_2 \nearrow_w \dots \searrow_w \bar{b}_{n-1}/\bar{a}_{n-1} \nearrow_w \bar{q}/\bar{p},$$

where $a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1} \in L$. Then there exists $a_1' \in \bar{a}_1$ such that $a \leqslant a_1'$. Letting $b_1' = a_1' + b$, we have $b_1' \in \bar{b}_1$ and $b/a \nearrow_w b_1'/a_1'$. Next, there exists $b_2' \in \bar{b}_2$ such that $b_2' \leqslant b_1'$. Letting $a_2' = b_2' a_1'$, we have $a_2' \in \bar{a}_2'$ and $b_1'/a_1' \searrow_w b_2'/a_2'$. Continuing this procedure we get

$$b/a \nearrow_w b_1'/a_1' \searrow_w b_2'/a_2' \nearrow_w \dots \searrow_w b'_{n-1}/a'_{n-1} \nearrow_w q'/p',$$

where $\bar{p}' = \bar{p}$ and $\bar{q}' = \bar{q}$. By the first argument, $q'/p' \nearrow_w (q'+q)/(p'+q) \searrow_w q/p$. Hence b/a projects weakly onto q/p in $(n+1)$ steps.

3. Main Theorem

THEOREM 3.1. *If V and V' are finitely based lattice varieties such that $R((V \cap V')_{si}) = r$ for a nonnegative integer r and if $F_V(r+3)$ and $F_{V'}(r+3)$ are finite, then $V + V'$ is finitely based.*

Here, for any class K of lattices, K_{si} denotes the class of all subdirectly irreducible members of K.

Proof. Let V and V' be defined by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, relative to the variety of all lattices. We may assume that the inclusions $\alpha \leqslant \beta$ and $\gamma \leqslant \delta$ hold in every lattice. Letting K and K' be the classes of $(r+3)$-generated subdirectly irreducible lattices in V and V', respectively, we write $h = \max(R(K), R(K'))$. Since the proof is trivial when $h = 0$, that is, V and V' are distributive, we assume from now on that $h > 0$. Letting $U = (V + V')^{r+3}$, we are going to show that the condition $P(n)$ in Lemma 2.1 holds for $n = \max(2h+5, h+r+5)$. Since U is finitely based by Lemma 2.2, it follows that $V + V'$ is also finitely based.

Consider a lattice $L \in U$, and suppose there exist $u, u' \in {}^{\omega}L$ and $c, d \in L$ with $c < d$ such that both $\beta(u)/\alpha(u)$ and $\delta(u')/\gamma(u')$ project weakly onto d/c in m steps and m' steps, respectively. Assuming that the three quotients have been so chosen that $m + m'$ is as small as possible, we shall show that the assumption $m>\max\ (2h+5, h+r+5)$ leads to a contradiction.

There exists a sequence of quotients

$$\beta(u)/\alpha(u)=b_0/a_0, b_1/a_1, \dots, b_m/a_m=d/c$$

such that, for successive values $i=0, 1, \dots, m-1$, b_i/a_i transposes weakly alternatingly up and down onto b_{i+1}/a_{i+1}. Choose some integer k so that $\max\ (h+2,\ r+2)<k<m-h-2$. Since the other cases can be treated similarly, we may assume that

$$b_{k-r-2}/a_{k-r-2}\nearrow_w b_{k-r-1}/a_{k-r-1}\searrow_w\dots\nearrow_w b_k/a_k.$$

Let $a_k=x_1$, $b_{k-1}=x_2, \dots, b_{k-r-2}=x_{r+3}$, and let L_0 be the sublattice of L generated by $x_1, x_2, \dots, x_{r+3}$. Observe that $F_{V+V'}(r+3)$ is finite and so is L_0. Hence $(x_1+x_2)/x_1=b_k/a_k$ can be divided into prime quotiens in L_0, z_1/z_0, z_2/z_1, $\dots$, z_s/z_{s-1} for a positive integer s, where $a_k=z_0\prec z_1 \prec\dots\prec z_s=b_k$. Then at least one of these quotients in L must projects weakly onto a nontrivial subquotient of d/c. Denote one such quotient by q/p.

By Lemma 2.3, we have a unique subdirectly irreducible quotient lattice $\bar{L}_0$ of L_0 in which $\bar{q}/\bar{p}$ is a critical quotient. Since $L_0\in \boldsymbol{V}+\boldsymbol{V}'$, $\bar{L}\in \boldsymbol{V}+\boldsymbol{V}'$. Therefore we have the following three cases.

Case (i). $\bar{L}_0\in \boldsymbol{V}$ but $\bar{L}_0\notin \boldsymbol{V}'$.

Since $\bar{L}_0\notin \boldsymbol{V}'$, $\gamma(\bar{v})<\delta(\bar{v})$ for some $v\in{}^\omega L_0$. Observe that $\gamma(\bar{v})=\overline{\gamma(v)}$ and $\delta(\bar{v})=\overline{\delta(v)}$. Since $\bar{L}_0\in \boldsymbol{V}$, $R(\bar{L}_0)\leqslant h$ and therefore, by Lemma 2.4, $\delta(v)/\gamma(v)$ projects weakly onto q/p in $(h+1)$ steps. But $\beta(u)/\alpha(u)$ projects weakly onto q/p in $(k+1)$ steps. Now this is a contradiction because $h+k+2<m\leqslant m+m'$.

Case (ii) $\bar{L}_0\in \boldsymbol{V}'$ but $\bar{L}_0\notin \boldsymbol{V}$.

Since $\bar{L}_0\notin \boldsymbol{V}$, $\alpha(v)<\beta(v)$ for some $v\in{}^\omega L_0$. Again, since $\bar{L}_0\in \boldsymbol{V}'$, $R(\bar{L}_0)\leqslant h$ and then by Lemma 2.4, $\beta(v)/\alpha(v)$ projects weakly onto q/p in $(h+1)$ steps. So $\beta(v)/\alpha(v)$ projects weakly onto a nontrivial subquotient d'/c' of d/c in $(h+1+m-k)$ steps. But $m+h+1-k<m-1$. Since $\delta(u')/\gamma(u')$ projects weakly onto d'/c' in $(m'+1)$ steps, we again have a contradiction.

Case (iii). $\bar{L}_0\in \boldsymbol{V}\cap \boldsymbol{V}'$.

We divide it into two subcases.

Subcase 1. Assume that $\boldsymbol{V}\cap \boldsymbol{V}'$ is not distributive. Since $\bar{b}_{k-r-2}/\bar{a}_{k-r-2}$ projects weakly onto $\bar{q}/\bar{p}$ in r steps, it follows from Lemma 2.4 that b_{k-r-2}/a_{k-r-2} projects weakly onto q/p in $(r+1)$ steps. So we have either

$$b_{k-r-2}/a_{k-r-2} \searrow_w b'_{k-r-1}/a'_{k-r-1} \nearrow_w \cdots \searrow_w b'_{k-2}/a'_{k-2} \nearrow_w q/p$$

or

$$b_{k-r-2}/a_{k-r-2} \nearrow_w b'_{k-r-1}/a'_{k-r-1} \searrow_w \cdots \nearrow_w b'_{k-2}/a'_{k-2} \searrow_w q/p,$$

for some quotients $b'_{k-r-1}/a'_{k-r-1}, \ldots, b'_{k-2}/a'_{k-2}$ in L. Since $b_{k-r-3}/a_{k-r-3} \searrow_w b_{k-r-2}$ and $b_k/a_k \searrow_w b_{k+1}/a_{k+1}$, it follows that $\beta(u)/\alpha(u)$ projects weakly onto a nontrivial subquotient d'/c' of d/c in $(m-2)$ steps. Also, $\delta(u')/\gamma(u')$ projects weakly onto d'/c' in $(m'+1)$ steps, which leads to a contradiction.

Subcase 2. Assume that $V \cap V'$ is distributive. We see that b_{k-2}/a_{k-2} transposes up and down onto q/p and also b_{k-2}/a_{k-2} transposes down and up onto q/p at the same time. Hence $\beta(u)/\alpha(u)$ projects weakly onto a nontrivial subquotient d'/c' of d/c in $(m-2)$ steps. Since again $\delta(u')/\gamma(u')$ projects weakly onto d'/c' in $(m'+1)$ steps, we have another contradiction.

Let $\boldsymbol{N}_5$ be the variety generated by N_5, the five-element nonmodular lattice. We now state Jónsson's result and some immediate consequences of the above theorem.

COROLLARY 3.2 (Jónsson [3]). *If $\boldsymbol{V}$ is a finitely based modular lattice variety, and if $\boldsymbol{V}'$ is a finitely based lattice variety such that $(\boldsymbol{V}')^3 = (\boldsymbol{N}_5)^3$, then $\boldsymbol{V}+\boldsymbol{V}'$ is finitely based.*

COROLLARY 3.3 *If $\boldsymbol{V}$ is a finitely based modular lattice variety and if $\boldsymbol{V}'$ is a finitely based locally finite lattice variety excluding M_3, the five-element nondistributive modular lattice, then $\boldsymbol{V}+\boldsymbol{V}'$ is finitely based.*

COROLLARY 3.4. *If $\boldsymbol{V}$ and $\boldsymbol{V}'$ are finitely based locally finite lattice varieties such that $R(\boldsymbol{V} \cap \boldsymbol{V}')$ is finite, then $\boldsymbol{V}+\boldsymbol{V}'$ is finitely based.*

4. Near- and Almost Distributive Lattice Varieties

A lattice L is said to be *semidistributive* if, for all u, v, x, y, z in L, $u=xy=xz$ implies $u=x(y+z)$, and dually, $v=x+y=x+z$ implies $v=x+yz$, and a lattice variety is said to be *semidistributive* if each of its members is semidistributive. A lattice, or a lattice variety, is said to be *near-distributive* if it satisfies the identity

(ND) $\quad x(y+z)=x(y+x(z+xy))$, and its dual

(ND′) $$x+yz=x+y(x+z(x+y)).$$

Finally, a lattice, or a lattice variety, is said to be *almost distributive* if it is near-distributive and satisfies the identity

(AD) $$v(u+c)\leqslant u+c(v+a),$$

where $a=xy+xz$ and $c=x(y+xz)$, and its dual

(AD′) $$v+uc'\geqslant u(c'+va'),$$

where $a'=(x+y)(x+z)$ and $c'=x+y(x+z)$. (For more about near- and almost distributive lattice varieties, see Lee [6].)

In [3], Day introduced the notion of splitting of a quotient in a lattice. For a lattice L and a quotient I in L, the new lattice is the set $L[I]=(L-I)\cup(I\times 2)$ with the inclusion relation defined in an obvious manner: For $a,b\in L-I$, $c,d\in I$ and $i,j\in\{0,1\}$,

$a\leqslant b$ in $L[I]$ if and only if a b in L,
$a\leqslant(d,j)$ if and only if $a\leqslant d$,
$(c,i)\leqslant b$ if and only if $c\leqslant b$,
$(c,i)\leqslant(d,j)$ if and only if $c\leqslant$ d and $i\leqslant j$.

It is easy to check that $L[I]$ is indeed a lattice, and that there is an obvious epimorphism from $L[I]$ to L. When $I=\{a\}$ for some $a\in L$, we write $L[I]=L[a]$.

Our next theorem is an application of Theorem 3.1.

THEOREM 4.1. *The join of two finitely based almost distributive lattice varieties is also finitely based.*

Proof. By Corollary 3.4, it suffices to show that $R(\boldsymbol{V}_{si})\leqslant 3$ for any almost distributive lattic variety $\boldsymbol{V}$, since $\boldsymbol{V}$ is locally finite. In fact, if $L\in\boldsymbol{V}_{si}$ then $L\cong D[d]$ for some distributive lattice D and $d\in D$ (Corollary 3.6 [6]). We now consider $D[d]$.

Suppose v/u projects weakly onto q/p in $D[d]$. Then there exists a sequence of quotients

$$v/u=b_0/a_0, b_1/a_1, \ldots, b_n/a_n=q/p$$

such that, for successive values $i=0,1,\ldots,n-1$, b_i/a_i transposes weakly alternatingly up and down onto b_{i+1}/a_{i+1}. Now we consider two cases.

Case 1. $(d,1)/(d,0)\nsubseteq q/p$.

Let $\bar{D}$ be the obvious quotient lattice of $D[d]$, i.e., identifying only $(d,0)$ and $(d,1)$. In $\bar{D}$, we have $\bar{v}/\bar{u}\nearrow_w\bar{b}/\bar{a}\searrow_w\bar{q}/\bar{p}$ for some a, $b\in D[d]$

Choose $a'\in\bar{a}$ with $a'\geqslant u$. Letting $b'=a'+v$, we have $b'\in\bar{b}$ and $v/u\nearrow_w b'/a'$. Similarly $b'/a'\searrow_w q'/p'$ for some $p'\in\bar{p}$ and $q'\in\bar{q}$. The homomorphism $x\rightarrow\bar{x}$ is one-to-one, except that $(d,0)$ and $(d,1)$ go into the same element. Consequently, we have $b'/a'\searrow_w q/p$, unless one of the elements p and q is equal to $(d,0)$ or $(d,1)$. The possibilities $p=(d,0)$ and $q=(d,1)$ are excluded because of the hypothesis of the present case. If $p=(d,1)$ and $p'\neq p$, then $p'=(d,0)$ and $q=q'$. Since $(d,0)$ is meet irreducible, we have $q'/p'\subseteq b'/a'$, hence $q/p\subseteq b'/a'$, whence it follows that $v/u\nearrow_w b'/p\searrow_w a/p$. If $q=(d,0)$ and $q'\neq q$, then $q'=(d,1)$ and $p=p'$, so that $v/u\nearrow_w b'/a'\searrow_w q/p$. We now conclude that v/u transposes weakly up and down (and dually, down and up) onto q/p.

Case 2. $(d,1)/(d,0)\subseteq q/p$.

Let k be the smallest integer such that $(d,1)/(d,0)\subseteq b_k/a_k$. When $k=0$, v/u projects weakly onto a nontrivial subquotient $(d,1)/(d,0)$ of q/p in one step. When $k>0$, it follows from Case 1 that v/u projects weakly onto b_{k-1}/a_{k-1}, and therefore onto b_k/a_k, in two steps, so that v/u projects weakly onto a nontrivial subquotient $(d,1)/(d,0)$ of q/p in three steps.

However, it is known in [6] that the variety of all near-distributive lattices is not locally finite. Considering the near-distributive lattice J_n (Figure 1), we now can show that the join of two finitely based near-distributive lattice varieties need not be finitely based. In fact, it can be easily checked that J_n is bounded (Corollary 5.4 [4]), which implies that J_n is semidistributive (Corollary 5.3 [7]). In addition, since J_n does not contain L_{11} and L_{12} (Figure 1) as sublattices, it follows from Corollary 2.6 in [6] that J_n is near-distributive. Now consider two finitely based near-distributive lattice varieties V and V' which are the conjugate varieties of splitting lattices L_{13} and L_{14} (Figure 1), respectively, relative to the variety of all near-distributive lattices. The rest of the proof is quite tedious and we omit it (cf. Theorem 1.1 [5]).

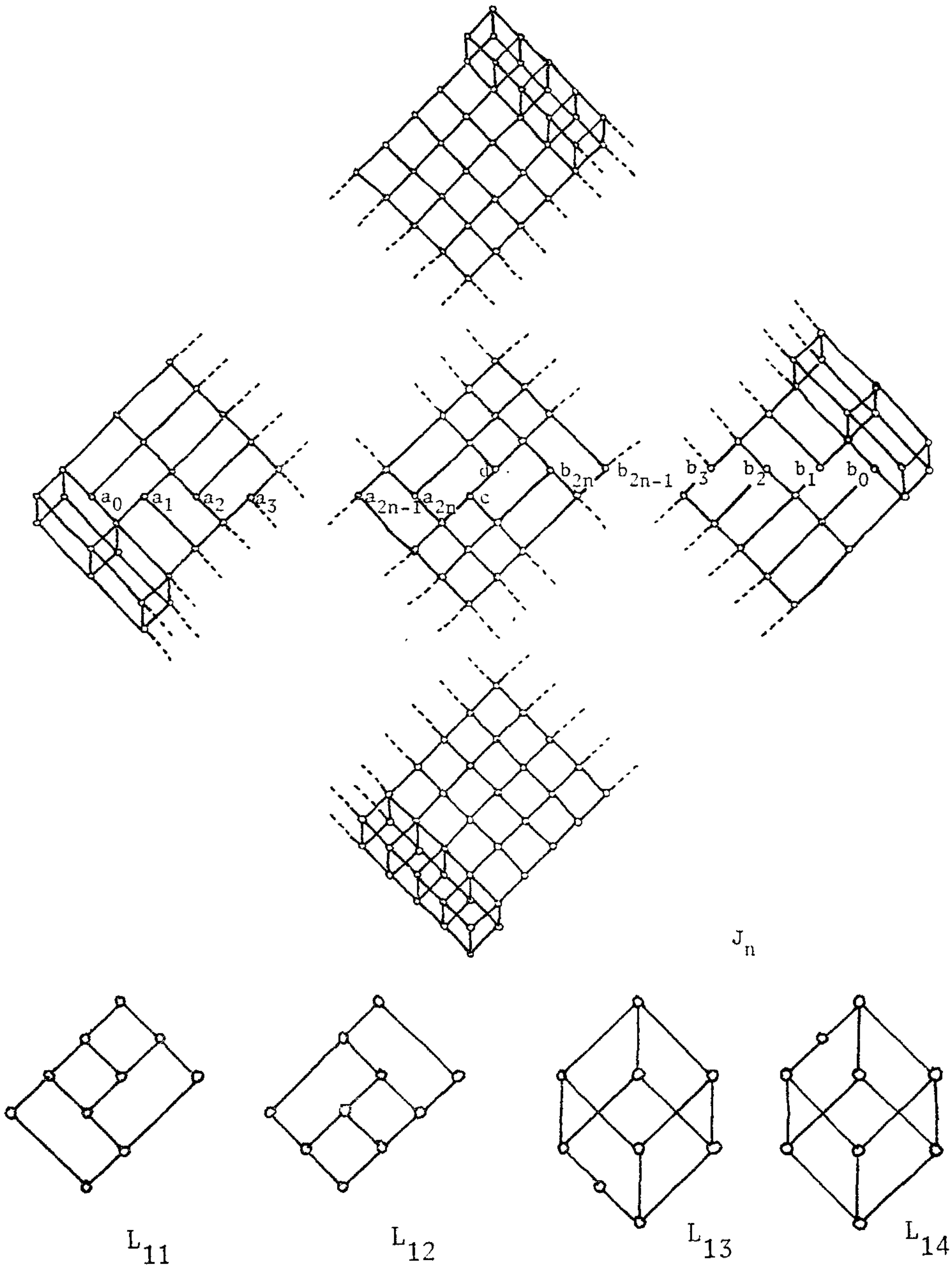

Figure 1

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