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INTRINSIC STRUCTURES IN AN ORDERED VECTOR SPACE*

KYUNG CHAN MIN

Introduction

On a partially ordered vector space, there are various intrinsic structures, in the sense that they are determined by the order structure. In particular, order convergence, relative uniform convergence and order boundedness have long been studied with extensive applications in the theory of ordered topological vector spaces. The concept of order convergence has been introduced by G. Birkhoff [3] and L. V. Kantorovitch [12]. The concept of relative uniform convergence goes back to E. H. Moore [15]. The order bornology, the bornology generated by order bounded sets, has been introduced by M-T. Akkar [1]. In developing the theory of ordered topological vector spaces, interrelations among these intrinsic structures have been played essential roles.

The purpose of this paper is to display intimate relationships among these intrinsic structures in an ordered vector space by putting them into suitable settings. Following a preliminary section, we show order convergence and relative uniform convergence induce subcategories of the category of convergence vector spaces and investigate standard constructions in these categories. The notion of equivalence of order convergence and relative uniform convergence was introduced by A. G. Pinsker [cf. 13] and was employed by W. A. J. Luxemburg and A. C. Zaanen [14], who say that in this case order convergence is stable. In §2, we generalize the notion of stability of order convergence and examine permanence properties. The stability of order convergence is an important necessary condition for order convergene to be topological. In §3, we show that order bounded sets also induce a subcategory of

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the category of bornological vector spaces. As an important result, the category of relative uniform convergence vector spaces is shown to be isomorphic to the category of order bound bornological vector spaces with a generating positive cone.

For ordered vector space theory we generally follow the terminology of Y-C. Wong and K-F. Ng [19] and for general categorical background we refer to H. Herrlich and G. E. Strecker [9].

0. Preliminaries.

CvV will denote the category of convergence spaces [2] (=limit vector spaces in [8]) and continuous linear maps. The subcategory determined by all locally convex convergence spaces in **CvV** will be denoted by **CCvV**. **BoV** will denote the category of bornological spaces [10] and bounded linear maps. The subcategory determined by all convex bornological spaces in **BoV** will be denoted by **CBoV**.

We introduce some intimate relationships among convergence vector spaces, locally convex spaces and bornological vector spaces.

A functor B : **CvV** $\to$ **BoV** (called the *von-Neumann functor*) is determined by $B(E, \Lambda)=(E, \mathscr{B}_\Lambda)$ and $B(f)=f$, where $\mathscr{B}_\Lambda=\{B\subseteq E : \mathscr{U}\cdot B\in\Lambda(0)\}$, $\mathscr{U}$ is the neighborhood filter at 0 in R and a functor K : **BoV**$\to$**CvV** is defined by $K(E, \mathscr{B})=(E, \Lambda_\mathscr{B})$ and $K(f)=f$, where $\Lambda_\mathscr{B}(0)=$the set of all filters on E containing $\mathscr{U}\cdot B$ for some $B\in\mathscr{B}$ and $\Lambda_\mathscr{B}(x)=\Lambda_\mathscr{B}(0)+x$ for each $x\in E$. The restrictions of B and K between **CCvV** and **CBoV** are functors. We will use the same notations B and K for the restriction of the functors B and K, respectively. It is well-known [5] that the functor K is a left adjoint of the functor B.

The category **LCV** is formed by all locally convex spaces and continuous linear maps. A functor B : **LCV**$\to$**CBoV** is defined by $BE=(E, \mathscr{B}_t)$ and $B(f)=f$, where $\mathscr{B}_t$=the collection of bounded subsets of E (called the ***von-Neumann bornology*** on E), and a functor T : **CBoV**$\to$ **LCV** is defined by $T(E, \mathscr{B})=(E, \mathscr{U}_t)$ and $T(f)=f$, where $\mathscr{U}_t$ is the neighborhood filter at 0 generated by the filter base $\mathrm{co}(\bigcap_{B\in\mathscr{B}}\mathscr{U}\cdot B)=$ {convex hull of $\bigcup_{B\in\mathscr{B}}(-\varepsilon_B, \varepsilon_B)\cdot B : \varepsilon_B>0\}$. The functor T is a left adjoint of B (cf. [10]). The category **LCV** is a bireflective subcategory of the category **CCvV** (cf. [8]).

A locally convex space E is *bornological* [10] iff $E=TBE$. For every $E\in\mathbf{CBoV}$, the locally convex space TE is bornological.

1. Intrinsic convergences.

In this section, we recall two intrinsic convergences, order convergence and relative uniform convergence, in an ordered vector space, which will induce subcategories of the category CvV. We will investigate the behavior of intrinsic convergences in subspaces, products and coproducts in the category OV of ordered vector spaces.

A. Order convergence.

We show that order convergence induces a subcategory of CvV.

1.1. Definition: (cf. [7]) Let E be an ordered vector space. A subset A of E is said to be *down directed to* 0 in E if for each $x, y\in A$ there exists $z\in A$ such that $z\leqslant x$ and $z\leqslant y$, and inf $A=0$ in E. A filter $\mathfrak{F}$ on E is said to be *order convergent* to 0 if $\mathfrak{F}\supseteq\mathfrak{F}_E(A)$ (or simply $\mathfrak{F}(A)$) for some subset A of E which is down directed to 0, where $\mathfrak{F}_E(A)$=the filter on E generated by the family $\{[-a, a] : a\in A\}$. We denote this convergence by "$\xrightarrow[E]{o}$" or simply "$\xrightarrow{o}$".

Remark: Our definition of order convergence is equivalent to A. J. Ward's [18]: A filter $\mathfrak{F}$ is "order convergent" to 0 in E if there exists a filter $\mathcal{G}$ with $\mathfrak{F}\supseteq\mathcal{G}$ such that sup $L(\mathcal{G})=$inf $U(\mathcal{G})=0$, where $L(\mathcal{G})=\{x\in E : x\leqslant G$ for some $G\in\mathcal{G}\}$ and $U(\mathcal{G})$ is defined dually. (Cf. 2.5 in [16])

An ordered vector space E is said to be *Archimedean* if $x\leqslant 0$ whenever $nx\leqslant y$ for all $n\in N$ and some $y\in E$

Proposition (3.9, [16]): *Let E be an ordered vector space, $\Lambda^\circ(0)=$ the set of all order convergent filters to 0 in E and $\Lambda^\circ(x)=\Lambda^\circ(0)+x$ for each $x\in E$. If E is Archimedean and has a generating positive cone, i.e. $E=C-C$, then $\Lambda^\circ=\{\Lambda^\circ(x) : x\in E\}$ is a locally convex convergence structure on E. (We will refer to (E, Λ°) as an order convergence vector space and denote simply by (E, o)).*

Remark. Since the scalar multiplication is continuous, it is easy to see that the condition "E is Archimedean and $E=C-C$" are necessary for Λ° to be a convergence vector structure on E.

1.2. DEFINITION: Let E and F be Archimedean ordered vector spaces with a generating positive cone. A linear map $f:(E, o)\to(F, o)$ is called *o-continuous* if $f(\Lambda_E^o(0))\subseteq\Lambda_F^o(0)$. The category OC is formed by all order convergence vector spaces and all o-continuous linear maps.

PROPOSITION: *OC is a subcategory of CvV.*

Now we examine the behavior of order convergence in subspaces, products and coprducts in the category OV.

1.3. A subspace M of an ordered vector space E with a generating positive cone does not have to have a generating positive cone. Thus, in general, (M, Λ°) is not an order convergence vector space.

PROPOSITION: *Let E be an order complete vector lattice and M a band in E. Then (M, o) is a subspace of (E, o) in CvV.*

Proof. Let $\mathcal{F}$ be a filter on M. Suppose $\mathcal{F}\xrightarrow[M]{o}0$ in M. Then there exists $A\subseteq M$ which is down directed to 0 in M such that $\mathcal{F}\supseteq\mathcal{F}_M(A)$. Since M is a solid subset of E $\inf A=0$ in E and hence A is down directed to 0 in E. Hence ci $(\mathcal{F})\supseteq\mathcal{F}_E(A)$ implies $\mathrm{ci}(\mathcal{F})\xrightarrow[E]{o}0$.

Conversely, suppose $\mathrm{ci}(\mathcal{F})\xrightarrow[E]{o}0$. Then there exists $A\subseteq E$ which is down directed to 0 in E such that $\mathrm{ci}(\mathcal{F})\supseteq\mathcal{F}_E(A)$. (We may assume that for each $F\in\mathcal{F}$, $0\in F$). Let $Q=\{F\in\mathcal{F}: F\subseteq[-a, a]$ for some $a\in A\}$ and let $x_F=\sup_E F$ and $y_F=\inf_E F$ for each $F\in Q$. Then for each $F\in Q$ $y_F\leqslant 0\leqslant x_F$, x_F and y_F is in M and hence $z_F=x_F\vee(-y_F)\in M$ since M is a band of E. Let $A^*=\{z_F: F\in Q\}$. Then A^* is down directed to 0 in M. Hence $\mathcal{F}\supseteq\mathcal{F}_M(A^*)$ implies $\mathcal{F}\xrightarrow[E]{o}0$.

1.4. Let $\{E_i\}_I$ be a family of Archimedean ordered vector spaces with a generating positive cone, where I is an index set. Then the product $\prod_I E_i$ in OV is also an Archimedean ordered vector space with a generating positive cone. Let pr_i^*: $(\prod_I E_i, o)\to(E_i, o)$ be the projection function for each $i\in I$. Then the linear map pr_i^* is continuous for each $i\in I$.

PROPOSITION (2.16, [16]): *If I is finite, then $((\prod_I E_i, o), (pr_i^*)_I)$ is the product of the family $\{(E_i, o)\}_I$ in CvV.*

REMARK: For an arbitrary index set I, $1: (\prod_I E_i, o) \to \prod_I (E_i, o)$ is continuous. However, in general, for an infinite index set I, $1: \prod_I (E_i, o) \to (\prod_I E_i, o)$ is not continuous. Let $E_i = R$ for each $i \in I$. Then the neighborhood filter $\mathscr{U}_0$ at 0 in $\prod_I (R, \mathrm{o})$ is not order convergent to 0. Note that (R, o) has the usual topology.

1.5. Coproducts in **OV** behave nicely.

Let $\{E_i\}_I$ be a family of Archimedean ordered vector spaces with a generating positive cone, where I is an arbitrary index set. Then the coroduct $\bigoplus_I E_i$ in **OV** is also an Archimedean ordered vector space with a generating positive cone. Let $e_i^*: (E_i, \mathrm{o}) \to (\bigoplus_I E_i, \mathrm{o})$ be the canonical injection. Then the linear map e_i^* is continuous.

PROPOSITION: *For an arbitrary index set I, $((\bigoplus_I E_i, o), (e_i^*)_I)$ is the coproduct of the family $\{(E_i, o)\}_I$ in* **CvV**.

Proof. Since e_i^* is a continuous linear map for each $i \in I$, $1: \coprod_I (E_i, \mathrm{o}) \to (\bigoplus_I E_i, \mathrm{o})$ is continuous by the co-universal property of the coproduct $\coprod_I (E_i, \mathrm{o})$ in **CvV**. Conversely, let $\mathfrak{F} \xrightarrow{o} 0$ in $\bigoplus_I E_i$. Then there exists $A \subseteq \bigoplus_I E_i$ which is down directed to 0 such that $\mathfrak{F} \supseteq \mathfrak{F}(A)$. Take any $a \in A$ and $F_a \in \mathfrak{F}$ with $F_a \subseteq [-a, a]$. Then there exists a finite subset J of I such that $a_i = 0$ for all $i \in I \backslash J$. Therefore $e_J \circ pr_J(F_a) = F_a$, where pr_J is the projection onto $\prod_J E_i$. Observe that $pr_J(A)$ is down directed to 0 in $\prod_J E_i$ and $pr_J(\mathfrak{F}) \supseteq \mathfrak{F}(pr_J(A))$ and hence $pr_J(F) \xrightarrow{o} 0$ in $\prod_J E_i$. For each $F \in \mathfrak{F}$, $e_J \circ pr_J(F) \supseteq F \cap F_a$ and hence $\mathfrak{F} \supseteq e_J(pr_J(\mathfrak{F}))$, i.e. $\mathfrak{F}$ converges to 0 in $\coprod_I (E_i, \mathrm{o})$.

B. Relative uniform convergence.

We introduce relative uniform convergence in an ordered vector space with a filter formation and study a related subcategory of **CvV**.

1.6. DEFINITION: Let (E, C) be an ordered vector space. A filter $\mathfrak{F}$ on E is said to be ***relative uniform convergent*** to 0 if $\mathfrak{F} \supseteq \mathfrak{F}_E(a)$ (or simply $\mathfrak{F}(a)$) for some $a \in C$, where $\mathfrak{F}_E(a)$ = the filter on E generated by the family $\{[-n^{-1}a, n^{-1}a] : n \in N\}$. We denote this convergence by "$\xrightarrow[E]{r}$" or simply "$\xrightarrow{r}$".

REMARK: If E is Archimedean, $\mathfrak{F}\xrightarrow{r}0$ imlies $\mathfrak{F}\xrightarrow{\circ}0$, since $\{n^{-1}a\}_N$ is down directed to 0 in E. H. Nordman [16] introduced this notion, called "*equable order convergence*", under the restrictions, E is Archimedean and $E=C-C$ as a special kind of order convergence.

PROPOSITION: *Let E be an ordered vector space, $\Lambda^r(0)=$the set of all relative uniform convergent filters to 0 in E and $\Lambda^r(x)=\Lambda^r(0)+x$ for each $x\in E$ If $E=C-C$, then $\Lambda^r=\{\Lambda^r(x) : x\in E\}$ is a locally convex convergence vector structure on E. (We will refer to (E, Λ^r) as a relative uniform convergence vector space and denote by (E, r)).*

Proof. By routine work, it is easy to see that Λ^r is a convergence structure on E and $+ : (E, \Lambda^r)\times(E, \Lambda^r)\to(E, \Lambda^r)$ is continuous. For any $x\in E$, $\mathcal{U}\cdot x\xrightarrow{r} 0$, since $\mathcal{U}\cdot x\supseteq\mathfrak{F}(x_1+x_2)$, where $x=x_1-x_2$, $x_1, x_2\in C$. Moreover, for any $a\in C$, $(\mathfrak{F}(a)+\mathcal{U}\cdot x)+\alpha x\subseteq(\mathcal{U}+\alpha)\cdot(\mathfrak{F}(a)+x)$. Hence $\cdot : R\times(E, \Lambda^r)\to(E, \Lambda^r)$ is continuous.

REMARKS: The condition "$E=C-C$" is necessary for Λ^r to be a convergence vector structure on E, since the scalar multiplication is continuous. If E is Archimedean, then C is closed: Let $x\in C$. Then there exists $\mathfrak{F}\in\Lambda^r(x)$ such that $F\cap G\neq\phi$ for all $F\in\mathfrak{F}$. Since $\mathfrak{F}-x\supseteq\mathfrak{F}(a)$ for some $a\in C$, $-x\leqslant n^{-1}$ a for all $n\in N$ and hence $-x\leqslant 0$, i.e. $x\in C$. For an ordered vector space E with $E=C-C$, (E, r) is topological if and only if E has an order-unit. (cf. 2.2, [17]) Hence we can show that different vector orders on a vector space may induce the same relative uniform convergence: In R^2, let $C_1=$the usual positive cone and $C_2=\{(x, y)\in C_1 : y\leqslant x\}$. Then $E_1=(R^2, C_1)$ and $E_2=(R^2, C_2)$ have order-units and hence (E_1, r) and (E_2, r) are topological. Moreover, since C_1 and C_2 induce Archimedean orders, (E_1, r) and (E_2, r) are Hausdorff. Therefore $(E_1, r)=(E_2, r)$, while $C_2\subsetneq C_1$.

1.7. DEFINITION: Let E and F be ordered vector spaces with a generating positive cone. A linear map $f : (E, r)\to(F, r)$ is called *r-continuous* if $f(\Lambda_E^r(0))\subseteq\Lambda_F^r(0)$.

The category **RU** is formed by all relative uniform convergence vector spaces and r-continuous linear maps.

PROPOSITION: **RU** *is a subcategory of* **CvV**.

REMARK: Cleary, every positive linear map $f : (E, r)\to(F, r)$ is

r-continuous. However a positive linear map $f:(E, o)\to(F, o)$ is not necessarily o-continuous: Consider the following sequence $\{f_n\}_N$ in the space $C[0, 1]$, of all real-valued continuous functions on $[0, 1]$, with the natural order; For each $n\in N$, $f_n(x)=1$ on $[0, n^{-1}]$, linear on $[n^{-1}$ $n^{-2}]$, and 0 on $[n^{-2}, 1]$. Then the sequence $\{f_n\}_N$ is down directed to 0. Define a map $H: C[0, 1]\to R$ by $H(f_n)=f_n(0)$. Then H is a positive linear map, while it is not o-continuous, since $H(f_n)=1$ for all $n\in N$.

Now we examine the behavior of relative uniform convergence in subspaces, products and coproducts in the category **OV**.

1.8. PROPOSITION: *Let E be an ordered vector space with $E=C-C$ and M a confinal subspace of E, i.e. for each $x\in C$, there is $y\in F$ such that $x\leqslant y$. Then (M, r) is a subspace of (E, r) in* **CvV**.

Proof. The canonical injection ci: $(M, r)\to(E, r)$ is continuous, since ci is positive. Let $\mathfrak{F}$ be a filter on M such that $ci(\mathfrak{F})\xrightarrow[E]{r} 0$. Then there exists $a\in C$ such that $ci(\mathfrak{F})\supseteq\mathfrak{F}_E(a)$. Since M is a cofinal subspace of E, there exists $b\in M$ such that $a\leqslant b$ and hence $\mathfrak{F}_E(a)\supseteq\mathfrak{F}_E(b)$. Thus $\mathfrak{F}\supseteq\mathfrak{F}_M(b)$, i.e. $\mathfrak{F}\xrightarrow[M]{r} 0$.

1.9. Let $\{E_i\}_I$ be a family of ordered vector spaces with a generating positive cone, where I is an index set. Let pr_i^*: $(\prod_I E_i, r)\to(E_i, r)$ be the projection function for each $i\in I$. Then the positive linear map pr_i^* is continuous for each $i\in I$.

PROPOSITION (3.14, [16]): *If I is finite, then $((\prod_I E_i, r), (pr_i^*)_I)$ is the product of the family $\{(E_i, r)\}_I$ in* **CvV**.

REMARK. For an arbitrary index set I, 1: $(\prod_I E_i, r)\to\prod_I(E_i, r)$ is continuous. However, in general, $1:\prod_I(E_i, r)\to(\prod_I E_i, r)$ is not continuous for an infinite index set I by the same reasoning as in Remark 1.4.

1.10. Coproducts in **OV** also behave nicely.

Let $\{E_i\}_I$ be a family of ordered vector spaces with a generating positive cone, where I is an arbitrary index set.

Let $e_i^*:(E_i, r)\to(\bigoplus_I E_i, r)$ be the canonical injection for each $i\in I$. Then the positive linear map e_i^* is continuous for all $i\in I$.

PROPOSITION: *For an arbitrary index set I, $((\bigoplus_I E_i, r), (e_i^*)_I)$ is the coproduct of the family $\{(E_i, r)\}_I$ in* CvV.

Proof. The proof is similar to that of Prop. 1.5.

2. Stability of order convergence.

We introduce the notion of stability of order convergent filters in an ordered vector space and examine some permanence properties.

2.1. In Luxemburg and Zaanen [14], order convergence for sequences in a vector lattice is said to be "stable" if for any sequence $x_n \xrightarrow{o} 0$ there exists a sequence $\{\lambda_n\}_N$ of real numbers such that $0 \leqslant \lambda_n \uparrow \infty$ and $\lambda_n f_n \xrightarrow{o} 0$. In R. Cristescu [6], a σ-complete vector lattice in which order convergence for sequences is stable is called a regular space. Indeed, in an Archimedean vector lattice order convergence is stable if and only if order convergence and relative uniform convergence for sequences are equivalent. We generalize this notion to order convergent filters.

DEFINITION: Order convergence in an ordered vector space is said to be *stable* if it is equivalent to relative uniform convergence.

REMARKS: If E is an Archimedean ordered vector space with $E=C-C$, then order convergence is stable if and only if $1 : (E, o) \to (E, r)$ is continuous. Hence order convergence in $C[0,1]$ with the usual order is not stable. (Consider the sequence $\{f_n\}_N$ mentioned in Remark 1.7) If order convergence in an ordered vector space E with $E=C-C$ is stable, then E must be Archimedean: Indeed, for each $a \in C$, $\mathfrak{F}(a) \supseteq \mathfrak{F}(A)$ for some $A \subseteq E$ which is down directed to 0 and hence $\inf_N n^{-1}a=0$.

2.2 H. Nordman (2.3, [17]) and S. Y. Xu (3, [20]) showed that for an Archimedean ordered vector space with $E=C-C$, (E, o) is topological if and only if E has an order-unit e and order convergence on E is stable. In this case, (E, o) is an order-unit normed space (1, [20]). Further, if E is order complete, then (E, o) is a Banach space since every order complete order-unit normed space is complete (3.7.1, 3.7.4, [11]). Thus it is interesting to check permanence properties of stability of order convergence.

2.3. PROPOSITION: *For a countable index set I, let $\{E_i\}_I$ be a family of Archimedean ordered vector spaces with a generating positive cone. If for every $i\in I$ order convergence on E_i is stable, then so is order convergence on the product $\prod_I E_i$ in* OV.

Proof. Let A be a down directed subset of $\prod_I E_i$ to 0. Then for each $i\in I$ $pr_i(A)$ is down directed to 0 in E_i and hence there exists $e_i\in CE_i$ such that $\mathfrak{F}(pr_i(A))\supseteq\mathfrak{F}(e_i)$. Take $a^*\in A$ and let $e=(e_1+a_1{}^*, 2(e_2+a_2{}^*), \ldots, i(e_i+a_i{}^*), \ldots)$. Fix $n\in N$. Then for each $k=1, \ldots, n$, there exists $a^k\in A$ such that $pr_k(a^k)\leqslant n^{-1}e_k$ and hence $pr_k(a^k)\leqslant n^{-1}k(e_k+a_k{}^*)$. Take $\hat{a}^n\in A$ such that $\hat{a}^n\leqslant a^1, a^2, \ldots, a^n$, a^* (A is down directed). Then $\hat{a}^n\leqslant n^{-1}e$. Therefore $\mathfrak{F}(A)\supseteq\mathfrak{F}(e)$ in $\prod_I E_i$.

REMARK: In general, this Proposition is not true for a uncountable index set I.

COUNTEREXAMPLE: Consider $\prod_I R$, where I is a uncountable index set. For each finite subset J of I and for each $m\in N$, we define a function $f_{J,m}: I\rightarrow R$ by $f_{J,m}(t)=m^{-1}$ if $t\in J$, and 1 otherwise. Then $A=\{f_{J,m}$: J is a finite subset of I, $m\in N\}$ is down directed to 0. Suppose order convergence on $\prod_I R$ is stable, then there exists $e\geqslant 0$ in $\prod_I R$ such that $\mathfrak{F}(A)\supseteq\mathfrak{F}(e)$. Hence for each $n\in N$ there exists a finite subset J_n of I and $m(n)\in N$ such that $f_{J_n,m(n)}\leqslant n^{-1}e$ and therefore for each $t\in I\backslash J_n$, $n\cdot f_{J_n,m(n)}(t)=n\cdot 1=e(t)$. Observe that $I\backslash\bigcup_N J_n\neq\phi$. Thus for any $t\in I\backslash\bigcup_N J_n$ $n\leqslant e(t)$ for all $n\in N$, which is impossible.

2.4. Stability of order convergence on an ordered vector space is not inherited by a subspace in general.

COUNTEREXAMPLE: Let $X=\{1, 2^{-1}, 3^{-1}, \ldots, n^{-1}, \ldots\}\cup\{0\}$ be a subspace of $[0,1]$. Then the space $C(X)$ of all real-valued continuous functions on X with the usual order is an Archimedean ordered vector space with a generating positive cone. For each $n\in N$, let $f_n(x)=0$ $(n^{-1}\leqslant x\leqslant 1)$ and 1 $(0\leqslant x\leqslant (n+1)^{-1})$. Then the set $A=\{f_n\}_N$. is down directed to 0. However $\mathfrak{F}(A)$ does not relative uniform converge to 0 obviously. Thus order convergence on $C(X)$ is not stable, while R^X is stable by the above Proposition.

2.5. PROPOSITION: *For an arbitrary index set I, let $\{E_i\}_I$ be a family of Archimedean ordered vector spaces with a generating positive cone.*

If for each $i\in I$ order convergence on E_i is stable, then so is order convergence on the coproduct $\bigoplus_I E_i$ in **OV**.

Proof. Suppose a subset A of $\bigoplus_I E_i$ is down directed to 0. Fix $a\in A$ and let J be a finite subset of I such that $a_i=0$ for all $i\in I\backslash J$. Observe that $pr_J(A)$ is down directed to 0 in $\bigoplus_J E_i$ and hence $\mathcal{F}(pr_J(A))\supseteq\mathcal{F}(e)$ in $\bigoplus_J E_i$ for some $e\in C(\bigoplus_J E_i)$ by Prop. 2.3. Indeed, $\mathcal{F}(A)\supseteq\mathcal{F}(ci(e))$ in $\bigoplus_I E_i$, where $ci:\bigoplus_J E_i\to\bigoplus_I E_i$ is the canonical injection: For each $n\in N$ there exists $a^n\in A$ such that $[-pr_J(a^n), pr_J(a^n)]\subseteq[-n^{-1}e, n^{-1}e]$. Take $\hat{a}^n\in A$ such that $\hat{a}^n\leqslant a^n$, a. Then $[-\hat{a}^n, \hat{a}^n]\subseteq[-n^{-1}ci(e), n^{-1}ci(e)]$.

3. Order-bounded sets and intrinsic structures.

In this section, we introduce a category **OBoV** of bornological vector spaces generated by order-bounded sets and investigate the images of **OBoV** under the functors T and K. It turns out that a subcategory of **OBoV** is isomorphic to the category **RU** of relative uniform convergence spaces.

3.1. PROPOSITION: *Let E be an ordered vector space and $\bar{\mathcal{B}}_E$ (or simply $\bar{\mathcal{B}}$)$=\{B\subseteq E : B\subset\sum_{i=1}^n bc([a_i, b_i]),\ a_i, b_i\in E,\ n\in N\}$, where $bc([a, b])$ is the balanced convex hull of $[a,b]$. Then $(E,\bar{\mathcal{B}}_E)$ is a convex bornological vector space. (We will refer to $(E,\bar{\mathcal{B}}_E)$ as an order bound bornological vector space.)*

Proof. Obviously, the family $\bar{\mathcal{B}}_E$ is a bornology on E and stable under vector addition and homothetic transformation. Let $B\in\bar{\mathcal{B}}_E$. Then $B\subseteq\sum_{i=1}^n bc([a_i, b_i])$ for some $a_i, b_i\in E$, $n\in N$, and hence $bc(B)\subseteq bc\ (\sum_{i=1}^n bc\ ([a_i, b_i]))=\sum_{i=1}^n bc[a_i, b_i]$. Thus $\bar{\mathcal{B}}_E$ is stable under the formation of balanced convex hulls.

REMARK. $\bar{\mathcal{B}}_E$ is the finest convex vector bornology on E for which every order-bounded subset of E is bounded.

3.2. DEFINITION: Let E and F be ordered vec1or paces. A linear map $f:(E,\bar{\mathcal{B}})\to(F,\bar{\mathcal{B}})$ is called *o-bounded* if it maps every order-bounded subset of E into an orPer-bounded subset of F.

The category **OBoV** is formed by all order bound bornological vector spaces and all o-bounded linear maps.

PROPOSITION: **OBoV** *is a subcategory of* **BoV**.

3.3. Now, we show that the objects in $T(\mathbf{OBoV})$ are the order bound locally convex spaces, which have been studied in ordered topological vector spaces. We recall that the order bound topology $\mathscr{T}_b$ is the finest locally convex topology on E for which every order-bounded subset of E is bounded and $(E, \mathscr{T}_b)$ is bornological.

PROPOSITION: *For an ordered vector space E, $T(E, \bar{\mathscr{B}})$ is the order bound locally convex space $(E, \mathscr{T}_b)$.*

Proof. Observe that $co(\bigcap_{B \in \bar{\mathscr{B}}} \mathscr{V} B)$ is a neighborhood filter base at 0 in $T(E, \bar{\mathscr{B}})$. Thus for every $[a, b]$ in E, $\mathscr{V}[a, b] \supseteq co\,(\bigcap_{B \in \bar{\mathscr{B}}} \mathscr{V} B)$ and hence $[a, b]$ is bounded in the locally convex space $T(E, \bar{\mathscr{B}})$. Therefore $1 : (E, \mathscr{T}_b) \to T(E, \bar{\mathscr{B}})$ is continuous. On the other hand, $1 : (E, \bar{\mathscr{B}}) \to B(E, \mathscr{T}_b)$ is bounded, since every order-bounded subset of E is bounded in $B(E, \mathscr{T}_b)$. Thus $T(1) = 1 : T(E, \bar{\mathscr{B}}) \to TB(E, \mathscr{T}_b) = (E, \mathscr{T}_b)$ is continuous.

3.4 Let E be an ordered vector space with $E = C - C$. Then the order bound bornology $\bar{\mathscr{B}}$ is generated by the family $\{[-a, a] : a \in C\}$. M-T. Akkar [1] introduced this notion, called "order bornology" in an ordered vector space with a generating positive cone, and investigated this bornology, in particular, on order completion and bornological completion.

3.5. Let **OBoVC** be the subcategory of **OBoV** generated by ordered vector spaces with a generating positive cone. Indeed, this category **OBoVC** is isomorphic to the category **RU**.

LEMMA: *Let E be an ordered vector space with $E = C - C$. Then for a subset B of E, B is order-bounded if and only if it is bounded in (E, r), i.e. $\mathscr{U} B \xrightarrow{r} 0$.*

Proof. B is bounded in $(E, r) \Longleftrightarrow \mathscr{U}B \xrightarrow{r} 0 \Longleftrightarrow \mathscr{U}B \supseteq \mathscr{I}(a) = \mathscr{U}[-a, a]$ for some $a \in C$, $\Longleftrightarrow B \subseteq \alpha[-a, a] = [-\alpha a, \alpha a]$ for some $\alpha \in R$, where $\mathscr{U}$ is the neighborhood filter at 0 in R.

THEOREM: **RU** *and* **OBoVC** *are isomorphic categories*

Proof. Observe that by Lemma for an ordered vector space E with

$E=C-C$, $B(E,r)=(E,\mathscr{B})$, where B : **CvV**$\to$**BoV** is the von-Neumann functor. Then the restriction $\bar{B}$: **RU**$\to$**OBoVC** of B is a functor. Indeed, the functor $\bar{B}$ is full and faithful, and the associated object function $\bar{B} : Ob(\mathbf{RU})\to Ob(\mathbf{OBoVC})$ is a bijection. $\bar{B}$ is *full*: Let $f : B(E,r)=(E,\mathscr{B}_E)\to(F,\mathscr{B}_F)=B(F,r)$ be a bounded linear map. Then for each $a\in CE$ $f(\mathscr{F}_E(a))=f(\mathscr{U}[-a, a]=\mathscr{U}f[-a, a]\supseteq\mathscr{U}[-b, b]=\mathscr{F}_F(b)$ for some $b\in CF$ since $f[-a, a]\in\mathscr{B}_F$, and hence $f(\mathscr{F}_E(a))\xrightarrow{r}0$ in(F, r). Therefore $f : (E, r)\to(F, r)$ is continuous and $B(f)=f$. Obviously, $\bar{B}$ is *faithful*. Moreover, $\bar{B} : Ob(\mathbf{RU})\to Ob(\mathbf{OBoVC})$ is a *bijection*: By the above observation, B is surjective. Let $B(E,r)=(E,\mathscr{B}_E)=(F, \mathscr{B}_F)=B(F, r)$. Then for each $a\in CE$, $\mathscr{F}_E(a)=\mathscr{U}[-a, a]=\mathscr{U}[-b, b]=\mathscr{F}_F(b)$ for some $b\in CF$, since $[-a, a]\in\mathscr{B}_E=\mathscr{B}_F$.

REMARK: In fact, $(E, r)=K(E, \mathscr{B})$, since $\mathscr{F}(a)=\mathscr{U}[-a, a]$ for each $a\in C$. Let $\bar{K}$ be the restriction of K to **OBoVC**. Then $\bar{B}\circ\bar{K}=1$ and $\bar{K}\circ\bar{B}=1$.

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Yonsei University
Seoul 120, Korea.