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ANTI-INVARIANT SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER SPACE FORMS

Dedicated to Professor Chin Myung Chung on his sixtieth birthday

U-HANG KI AND JIN SUK PAK

Recently, T. Kashiwada [1], S. Tachibana [4] and I. Vaisman [5] studied locally conformal Kähler manifolds and K. Matsumoto [2, 3] investigated some properties of CR-submanifolds of locally conformal Kähler manifolds.

The Hopf manifold is a typical examples of a locally conformal Kähler manifold which admits no Kähler metric ([4]).

In this paper, we shall study anti-invariant submanifolds of locally conformal Kähler manifolds and mainly prove the following theorems:

THEOREM 1. *Let N^n be an $n(>3)$-dimensional anti-invariant submanifold of a locally conformal Kähler space form $M^{2n}(H)$. If the associated vector field of the Lee form tangents to N^n and if the second fundamental tensors commute, then N^n is a conformally flat space.*

THEOREM 2. *Let N^n be an $n(>3)$-dimensional anti-invariant submanifold of a locally conformal Kähler space form $M^{2n}(H)$ such that the associated vector field of the Lee form is normal to N^n. If the second fundamental tensors commute, then N^n is a conformally flat space.*

THEOREM 3. *Let N^m be an $m(>3)$-dimensional anti-invariant submanifold of a locally conformal Kähler space form $M^{2n}(H)$. If N^m is totally umbilical, then N^m is a conformally flat space.*

1. Preliminaries

Let $M^{2n}(F_\mu{}^\lambda, g_{\mu\lambda}, \alpha_\lambda)$ be an $2n$-dimensional locally conformal Kähler manifold (an l. c. k-manifold). By its definition, at any point there exists a neighborhood in which a conformal metric $g^*=e^{-2\rho}g$ is Kaehlerian, that is, $\nabla^*(e^{-2\rho}F_\lambda{}^\mu)=0$, $d\rho=\alpha$, where ∇^* denotes the covariant

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differentiation with respect to g^*.

It is well known (cf. [1]) that a Hermitian manifold $M^{2n}(F_\lambda{}^\mu, g_{\mu\lambda})$ is a l. c. k-manifold if and only if there exists a global 1-form α satisfying

$$(1.1) \qquad \nabla_\nu F_{\mu\lambda} = -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu},$$

$$(1.2) \qquad \nabla_\mu \alpha_\lambda = \nabla_\lambda \alpha_\mu,$$

$$(1.3) \qquad \beta_\lambda = -\alpha_\mu F_\lambda{}^\mu$$

It is called a 1-form α the Lee form.

An l.c. k-manifold is called an l.c. k-space form if it has a constant holomorphic sectional curvature H. Then the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ of an l. c. k-space form $M^{2n}(H)$ with constant holomorphic sectional curvature H is given by (cf. [1])

$$(1.4) \qquad \begin{aligned} 4R_{\omega\nu\mu\lambda} = & H(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda} + F_{\omega\lambda}F_{\nu\mu} - F_{\omega\mu}F_{\nu\lambda} - 2F_{\omega\nu}F_{\mu\lambda}) \\ & + 3(P_{\omega\lambda}g_{\nu\mu} - P_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}P_{\nu\mu} - g_{\omega\mu}P_{\nu\lambda}) - \tilde{P}_{\omega\lambda}F_{\nu\mu} \\ & + \tilde{P}_{\omega\mu}F_{\nu\lambda} - F_{\omega\lambda}\tilde{P}_{\nu\mu} + F_{\omega\mu}\tilde{P}_{\nu\lambda} + 2(\tilde{P}_{\omega\nu}F_{\mu\lambda} + F_{\omega\nu}\tilde{P}_{\mu\lambda}), \end{aligned}$$

where

$$(1.5) \qquad P_{\mu\lambda} = -\nabla_\mu \alpha_\lambda - \alpha_\mu \alpha_\lambda + \frac{1}{2}\|\alpha\|^2 g_{\mu\lambda},$$

$$(1.6) \qquad \tilde{P}_{\mu\lambda} = -P_{\mu\lambda}F_\lambda{}^\nu.$$

2. Anti-invatiant submanifolds of l. c. k-manifolds

Let N^m be an m-dimensional manifold immersed in a $2n$-dimensional l. c. k-manifold $M^{2n}(F_\mu{}^\lambda, g_{\mu\lambda}, \alpha_\lambda)$. Since the discussion is local, we may assume, if it is necessary, that N^m is imbbeded in M^{2n}. If the manifold M^{2n} is covered by a system of coordinate neighborhood $\{\tilde{U}, y^\lambda\}$ and N^m is covered by a system of coordinate neighborhoods $\{U, x^i\}$, where, here and in the sequel the indices $\kappa, \nu, \mu, \lambda, \ldots$; $k, j, i, h, \ldots$run over the range $\{1, 2, \ldots, 2n\}$; $\{1, 2, \ldots, m\}$ respectively, then the submanifold N^m can be represented by $y^\kappa = y^\kappa(x^i)$. Here and in the sequel we identify vector fields in N^m with the images under the differential mapping.

We put

$$(2.1) \qquad B_i{}^\lambda = \partial_i y^\lambda \quad (\partial_i = \partial/\partial x^i)$$

and denote by $C_y{}^\lambda$ $2n-m$ mutually orthogonal unit vectors normal to N^m, where here and in the sequel the indices $x, y, z, \ldots$run over the range $\{1, 2, \ldots, 2n-m\}$. Then the metric tensor g_{ji} of N^m and that of normal bundle are respectively given by

$$g_{ji}=g_{\mu\lambda}B_{ji}{}^{\mu\lambda},\quad g_{yx}=g_{\mu\lambda}C_{yx}{}^{\mu\lambda},$$

where $B_{ji}{}^{\mu\lambda}=B_j{}^{\mu}B_i{}^{\lambda}$ and $C_{yx}{}^{\mu\lambda}=C_y{}^{\mu}C_x{}^{\lambda}$.

If the transform by $F_\lambda{}^\kappa$ of any vector tangent to N^m is orthogonal to N^m, we say that the submanifold N^m is anti-invariant in M^{2n}. Since the rank of $F_\lambda{}^\kappa$ is $2n$, we have $m\leqq n$.

For an anti-invariant submanifold N^m in M^{2n}, we have equations of the form

$$F_\lambda{}^\kappa B_i{}^\lambda= \qquad +f_i{}^x C_x{}^\kappa, \tag{2.2}$$

$$F_\lambda{}^\kappa C_y{}^\lambda=-f_y{}^i B_i{}^\kappa+f_y{}^x C_x{}^\kappa, \tag{2.3}$$

$$\alpha_\lambda=\alpha^i B_{i\lambda}+\alpha^x C_{x\lambda}, \tag{2.4}$$

$$\beta_\lambda=\beta^i B_{i\lambda}+\beta^x C_{x\lambda}, \tag{2.5}$$

where $B_{i\lambda}=B_i{}^\mu g_{\mu\lambda}$ and $C_{x\lambda}=C_x{}^\mu g_{\mu\lambda}$.

Using $F_{\mu\lambda}=-F_{\lambda\mu}, F_{\mu\lambda}=F_\lambda{}^\kappa g_{\kappa\lambda}$, we have from (2.2) and (2.3),

$$f_{iy}=f_{yi},\quad f_{yx}=-f_{xy}, \tag{2.6}$$

where $f_{iy}=f_i{}^z g_{zy}, f_{yi}=f_y{}^j g_{ji}$ and $f_{yx}=f_y{}^z g_{zx}$.

Applying F to (2.2)-(2.5) and using (1.3) and these equations, we find

$$\begin{cases} \text{(i) } f_i{}^x f_x{}^j=\delta_i^j, \quad \text{(ii) } f_i{}^x f_x{}^y=0, \\ \text{(iii) } f_x{}^i f_i{}^y-f_x{}^z f_z{}^y=\delta_x^y, \\ \text{(iv) } \beta_i=-f_i{}^x\alpha_x, \quad \text{(v) } \beta_x=f_x{}^i\alpha_i-f_x{}^y\alpha_y. \end{cases} \tag{2.7}$$

Differentiating (2.2)-(2.4) covariantly along N^m and using (1. 1), (1.2), (1.3), (2.7), equations of Gauss

$$\nabla_j B_i{}^\kappa=h_{ji}{}^x C_x{}^\kappa \tag{2.8}$$

and those of Weingarten

$$\nabla_j C_y{}^\kappa=-h_j{}^i{}_y B_i{}^\kappa, \tag{2.9}$$

where ∇_j denotes the operator of covariant differentiation along N^m and $h_{ji}{}^x$ and $h_j{}^i{}_y=h_{jk}{}^x g^{ki} g_{xy}$, $(g^{ki})=(g_{ki})^{-1}$, are the second fundamental tensors of N^m with respect to the normals $C_x{}^\kappa$, we find

$$\begin{cases} \text{(i) } -\delta_j^h\beta_i+g_{ji}\beta^h+h_j{}^h{}_x f_i{}^x-h_{ji}{}^x f_x{}^h=0, \\ \text{(ii) } \nabla_j f_i{}^x=g_{ji}\beta^x-f_j{}^x\alpha_i+h_{ji}{}^z f_z{}^x, \\ \text{(iii) } \nabla_j f_x{}^i=\delta_j^i\beta_x-f_{jx}\alpha^i-h_j{}^i{}_z f_x{}^z, \\ \text{(iv) } \nabla_j f_x{}^y=-f_j{}^y\alpha_x+f_{jx}\alpha^y-h_j{}^i{}_x f_i{}^y+h_{ji}{}^y f_x{}^i, \\ \text{(v) } \nabla_j\alpha_i=\nabla_i\alpha_j, \end{cases} \tag{2.10}$$

where $\alpha^i=\alpha_j g^{ji}$, $\beta^i=\beta_j g^{ji}$, $\alpha^x=\alpha_z g^{zx}$ and $\beta^x=\beta_z g^{zx}$.

On the other hand, the equations of Gauss, Codazzi and Ricci are respectively given by

$$(2.11)\qquad R_{kjih}=R_{\omega\mu\nu\lambda}B_{kjih}{}^{\omega\mu\nu\lambda}+h_{khx}h_{ji}{}^x-h_{jhx}h_{ki}{}^x,$$

$$(2.12)\qquad R_{\omega\mu\nu\lambda}B_{kji}{}^{\omega\mu\nu}C_y{}^\lambda=\nabla_k h_{jiy}-\nabla_j h_{kiy},$$

$$(2.13)\qquad R_{kjyx}=R_{\omega\mu\nu\lambda}B_{kj}{}^{\omega\mu}C_{yx}{}^{\nu\lambda}-(h_k{}^i{}_y h_{jix}-h_j{}^i{}_y h_{kix}),$$

where R_{kjih} and R_{kjyx} are covariant components of the curvature tensors of N^m and the normal bundle respectively, $B_{kjih}{}^{\omega\mu\nu\kappa}=B_k{}^\omega B_j{}^\mu B_i{}^\nu B_h{}^\kappa$ and $B_{kji}{}^{\omega\mu\nu}=B_k{}^\omega B_j{}^\mu B_i{}^\nu$.

Let N^m be an anti-invariant submanifold of an l. c. k-space form $M^{2n}(H)$. Then by using (1.4), (2.2) and (2.3) we find that the equations (2.11) and (2.13) of Gauss and Ricci reduce to respectively

$$(2.14)\qquad 4R_{kjih}=H(g_{kh}g_{ji}-g_{ki}g_{jh})+3(P_{kh}g_{ji}-P_{ki}g_{jh}+g_{kh}P_{ji}-g_{ki}P_{jh})+4(h_{kh}{}^x h_{jix}-h_{ki}{}^x h_{jhx}),$$

$$(2.15)\qquad 4R_{kjyx}=H(f_{kx}f_{jy}-f_{ky}f_{jx})-\tilde{P}_{kx}f_{jy}+\tilde{P}_{ky}f_{jx}-f_{kx}\tilde{P}_{jy}+f_{ky}\tilde{P}_{jx}-4(h_k{}^i{}_y h_{jix}-h_j{}^i{}_y h_{kix})+2\tilde{P}_{kj}f_{yx}$$

where we have put

$$(2.16)\qquad P_{ji}=P_{\mu\nu}B_{ji}{}^{\mu\nu},\quad \tilde{P}_{jx}=\tilde{P}_{\mu\nu}B_j{}^\mu C_x{}^\nu,\ \tilde{P}_{kj}=\tilde{P}_{\mu\nu}B_{kj}{}^{\mu\nu}.$$

3. Poof of Theorem 1

Let N^n be an n-dimensional anti-invariant submanifold of an l. c. k-space form $M^{2n}(H)$. Then from (2.7), (i) and (iii), we can easily see that

$$(3.1)\qquad f_y{}^x=0.$$

Suppose that the associated vector field α^κ of the Lee form α is tangent to N^n, that is, $\alpha^x=\alpha_y g^{yx}=0$. Then, from (2.7), (iv), (2.10), (iii) and (3.1), we have

$$(3.2)\qquad \nabla_j f_x{}^i=\delta_j^i f_x{}^h\alpha_h-f_{jx}\alpha^i.$$

Applying the operator ∇_k to (3.2) and using the Ricci identities, we have

$$-R_{kjx}{}^y f_y{}^i+R_{kjh}{}^i f_x{}^h=\delta_j^i(\nabla_k f_x{}^h)\alpha_h-\delta_k^i(\nabla_j f_x{}^h)\alpha_h+\delta_j^i f_x{}^h\nabla_k\alpha_h-\delta_k^i f_x{}^h\nabla_j\alpha_h-(\nabla_k f_{jx}-\nabla_j f_{kx})\alpha^i-f_{jx}\nabla_k\alpha^i+f_{kx}\nabla_j\alpha^i,$$

from which, transvecting with $f_1{}^x$ and using (2.7) and (2.10) with $\alpha_x=0$, we can easily obtain

(3.3) $$R_{kjli}=R_{kjyx}f_l^y f_i^x+g_{ji}(\nabla_k\alpha_l+\alpha_k\alpha_l-\|\alpha\|^2 g_{kl}) -g_{ki}(\nabla_j\alpha_l+\alpha_j\alpha_l-\|\alpha\|^2 g_{jl})+g_{kl}(\nabla_j\alpha_i+\alpha_j\alpha_i) -g_{jl}(\nabla_k\alpha_i+\alpha_k\alpha_i).$$

On the other hand, since $\tilde{P}_{kx}=P_{ki}f_x^i$, (2.15) implies

$$4R_{kjyx}f_l^y f_i^x=H(g_{ki}g_{jl}-g_{kl}g_{ji})-P_{ki}g_{jl}+P_{kl}g_{ji}-g_{ki}P_{jl} +g_{kl}P_{ji}-4(h_k{}^h{}_y h_{jhx}-h_j{}^h{}_y h_{khx})f_l^y f_i^x,$$

which and (3.3) yield

$$4R_{kjli}=(H+3\|\alpha\|^2)(g_{ki}g_{jl}-g_{kl}g_{ji}) +3\{g_{ji}(\nabla_k\alpha_l+\alpha_k\alpha_l)-g_{ki}(\nabla_j\alpha_l+\alpha_j\alpha_l) +g_{kl}(\nabla_j\alpha_i+\alpha_j\alpha_i)-g_{jl}(\nabla_k\alpha_i+\alpha_k\alpha_i)\} -4(h_k{}^h{}_y h_{jhx}-h_j{}^h{}_y h_{khx})f_l^y f_i^x$$

because $P_{ji}=-\nabla_j\alpha_i-\alpha_j\alpha_i+\frac{1}{2}\|\alpha\|^2 g_{ji}$. Hence we have

(3.4) $$R_{kjli}=g_{ji}L_{kl}-g_{ki}L_{jl}+g_{kl}L_{ji}-g_{jl}L_{ki} -(h_k{}^h{}_y h_{khx}-h_k{}^h{}_y h_{khx})f_l^y f_i^x,$$

where $L_{ji}=-\frac{1}{8}(H+3\|\alpha\|^2)g_{ji}+\frac{3}{4}(\nabla_j\alpha_i+\alpha_j\alpha_i)$.

If the second fundamental tensors commute, then from (3.4), we see that the submanifold N^n is conformally flat, provided that $n>3$, which completes the proof of Theorem 1.

4. Proof of Theorem 2

Let N^n be an n-dimensional anti-invariant submanifold of an l. c. k - space form $M^{2n}(H)$. Suppose that the associated vector field α^κ of the Lee form α is normal to N^n, that is, $\alpha^i=\alpha_j g^{ji}=0$. Then, from (2.7), (v), (2.10), (iii) and (3.1), we have

$$\nabla_j f_x^i=0,$$

which and the Ricci identities yield

$$R_{kjh}{}^i f_x^h=R_{kjx}{}^y f_y^i,$$

and consequently

(4.1) $$R_{kjli}=R_{kjyx}f_l^y f_i^x.$$

On the other hand, since $\tilde{P}_{kx}=P_{ki}f_x^i$ and

$$P_{ki}=h_{ki}{}^x\alpha_x+\frac{1}{2}\|\alpha\|^2 g_{ki},$$

(2.15) and (4.1) imply

$$4R_{kjli}=(H-\|\alpha\|^2)(g_{ki}g_{jl}-g_{kl}g_{ji})-h_{ki}{}^x\alpha_x g_{jl}$$
$$+h_{kl}{}^x\alpha_x g_{ji}-g_{ki}h_{jl}{}^x\alpha_x+g_{kl}h_{ji}{}^x\alpha_x$$
$$-4(h_k{}^h{}_y h_{jhx}-h_j{}^h{}_y h_{khx})f_l{}^y f_i{}^x,$$

or equivalently

$$4R_{kjli}=g_{ki}L_{jl}+g_{jl}L_{ki}-g_{kl}L_{ji}-g_{ji}L_{kl}$$
$$-4(h_k{}^h{}_y h_{jhx}-h_j{}^h{}_y h_{khx})f_l{}^y f_i{}^x. \tag{4.2}$$

If the second fundamental tensors commute, then (4. 2) gives that the submanifold N^n is conformally flat, provided that $n>3$. Thus we have Theorem 2.

5. Proof of Theorem 3

Let N^m be an m-dimensional anti-invariant submanifold of an l. c. k-space form $M^{2n}(H)$. Suppose that the submanifold N^m is totally umbilical, that is

$$h_{ji}{}^x=h^x g_{ji},\quad h^x=\frac{1}{m}g^{ji}h_{ji}{}^x.$$

Then the equation (2. 14) of Gauss implies

$$4R_{kjih}=g_{kh}L_{ji}+g_{ji}L_{kh}-g_{ki}L_{jh}-g_{jh}L_{ki},$$

where $L_{ji}=(2h^x h_x-\frac{1}{2}H)g_{ji}+3P_{ji}$. Hence N^m is conformally flat, provided $m>3$, which completes the proof of Theorem 3.

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Kyungpook University
Taegu 635, Korea