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DUALIZING COMPLEXES OF SOME 1-DIMENSIONAL LOCAL RINGS

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The object of this paper is to obtain a Dualzing Complexes of some subrings of $k[[x]]$ where k is an infinite field.

First we study their Local Cohomology [Th 2-1], [Th 4-3] and Multiplicity [Th 3-2].

Secondly we obtain their Dualizing Complexes in the concrete form [Th 4-2]. The rings we consider are of the form $A=k[[x^{e_1}, x^{e_2}, \dots x^{e_n}]] \subset R=k[[x]]$ where e_i is an integer such that $0<e_1<e_2<\dots<e_n$, $(e_1, e_2, \dots, e_n)=1$.

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1. Introduction

We shall recall the definitions and some properties of the Local Cohomology, the Dualizing Complex and the Matlis Duality which are used in this paper.

Let A be a noetherian local ring with maximal ideal m and M be a finitely generated A-module. Let

$$L_m(M)=\{x\in M \mid m^k x=0 \text{ for some } k\}.$$

Then L_m is a left exact functor.

Let $H_m{}^i$ be the i-th right derived functor of L_m, then

$$H_m{}^i(M)=\varinjlim_{\nu} \mathrm{Ext}_A{}^i(A/m^\nu, M). \quad [10, \text{ p. } 428]$$

This $H_m{}^i(M)$ is called the i-th Local Cohomology module of the A-module M. Also it can be defined in the usual way taking an injective resolution $I^\cdot$:

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$$H_m^i(M) = H^i(L_m(I^{\cdot})). \quad [9, \text{ p. } 76]$$

A Dualizing Complex for A is an injective complex $I^{\cdot}$ in $\mathcal{Y}_c^b(A)$ [that is, $I^{\cdot}$ is bounded and all its cohomology modules are finitely generated] with the following property: whenever $X^{\cdot}$ is a complex in $\mathcal{Y}_c^b(A)$ then

$$\theta^{\cdot}: X^{\cdot} \longrightarrow \mathrm{Hom}_A\ (\mathrm{Hom}_A(X^{\cdot}, I^{\cdot}), I^{\cdot})$$

is a quasi-isomorphism. [12, p. 372]

According to [12, p. 378], an injective complex $I^{\cdot} \in \mathcal{Y}_c^b(A)$ is a Dualizing Complex for A iff the morphism of complexes

$$\alpha^{\cdot}: A \longrightarrow \mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$$

is a quasi-isomorphism.

If $I^{\cdot}$ is a Dualizing Complex for A as in [13, p. 201] we define $t(m; I^{\cdot})$ to be the unique integer t for which

$$H^t(\mathrm{Hom}_A(k, I^{\cdot})) \neq 0 \text{ where } k = A/m.$$

Then m occurs only in I^t.

The injective envelope of $k = A/m$ over A is denoted by $E_A(k)$. Let D denote the Matlis Duality functor:

$$D(-) = \mathrm{Hom}_A(-,\ E_A(k)).$$

If A is a complete noetherian local ring and M is a finitely generated A-module then $DD(M) = M$. And if A is a noetherian local ring and M is an Artinian A-module then also $DD(M) = M$. [5, p. 528]

If M is a finitely generated A-module and $I^{\cdot}$ is a Dualizing Complex for A then all the cohomology modules of the complex $H_A(M, I^{\cdot})$ are finitely generated. [12, p. 375] So $D(H^i\ (\mathrm{Hom}_A(M, I^{\cdot})))$ is Artinian. And if M is a finitely generated A-module then the Local Cohomology modules $H_m^i(M)$ are Artinian. [4, p. 201]
There is some relation between these two Artinian A-modules.

PROPOSITION 1.1. *Let $I^{\cdot}$ be a arbitrary Dualizing Complex for A. Let $t = t(m; I^{\cdot})$ and M be a finitely generated A-module. Then*

$$H_m^i(M) \cong D(H^{t-i}(\mathrm{Hom}_A(M, I^{\cdot})))$$

for each $i \geqslant 0$. [2, p. 436]

Let $R = k[[x_1, x_2, \dots, x_n]]$ be a formal power series ring over a field

k and $E=k[x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}]$.
We can define the scalar product:

$$(x_1^{\alpha_1}\dots x_n^{\alpha_n})(x_1^{-\beta_1}\dots x_n^{-\beta_n})=\begin{cases}0 & \text{if some } \alpha_i>\beta_i\\ x_1^{-(\beta_1-\alpha_1)}\dots x_n^{-(\beta_n-\alpha_n)} & \text{otherwise.}\end{cases}$$

Then E is an R-module. Also we may consider k as an R-module. Then E is the injective envelope of k over R. [8, p. 292]

2. 1-dimensional local domain

Let A be a integral domain. Then the injective envelope of A is the quotient field of A.
If R is a 1-dimensional Gorenstein local domain then R has an injective resolution;

$$0\longrightarrow R\longrightarrow K\longrightarrow E_R(k)\longrightarrow 0$$

where K is the quotient field of R and $E_R(k)$ is the injective envelope of the residue field k over R. [1, p. 10]

Consider the subring $A=k[[x^{e_1}, x^{e_2}, \dots, x^{e_n}]]$ of $R=k[[x]]$, where $0<e_1<e_2<\dots<e_n$, $(e_1, e_2, \dots, e_n)=1$.
R is integral over A and the quotient field of them are equal because $(e_1, e_2, \dots e_n)=1$.
The injective envelope of k over R is

$$E_R(k)=k[x^{-1}]=kf_0+kf_1+kf_2+\cdots$$

where $f_i=1/x^{i+1}$ mod R.

THEOREM 2.1. *Let (A, m, k) be a 1-dimensional noetherian local domain. Then the Local Cohomology of A is $H_m^1(A)=K/A$ where K is the quotient field of R. If A is Gorenstein then $H_m^1(A)=E_A(k)$.*

Proof. Let

$$J^{\cdot}: 0\longrightarrow A\longrightarrow J^0\longrightarrow J^1\xrightarrow{\partial_1} J_2\longrightarrow\cdots$$

be a minimal injective resolution of A. Then

$$J^0=K,\ J^1=E_A(K/A).$$

Applying the Local Cohomology functor L_m:

$$L_m(J^{\cdot}): 0\longrightarrow L_m(K)\longrightarrow L_m(J^1)\xrightarrow{L_m(\partial_1)} L_m(J^2)\longrightarrow\cdots$$

It is clear from the definition that $L_m(K)=0$ and since A is 1-dimensional local domain any non-zero ideal is $\mathfrak{M}$-primary, hence $K/A=L_m(K/A)\subset L_m(J_1)$. So, $H_m{}^1(A)=H^1(L_m(J^\cdot))=\ker(L_m(\partial_1))=K/A$.

If A is Gorenstein then by [11, p. 185] $H_m{}^1(A)=E_A(k)$.

3. Multiplicity of $k[[x^{e_1}, x^{e_2}, \ldots x^{e_n}]]$

LEMMA 3.1. *Let $e_1, e_2, \ldots, e_n$ be positive integers such that $e_1<e_2<\ldots<e_n$, $(e_1, e_2, \ldots, e_n)=1$.*

Let S be the semi-group generated by $\{e_1, e_2, \ldots, e_n\}$. Then there is a large integer n such that S contains all integers larger than n.

Proof. Since $(e_1, e_2, \ldots e_n)=1$ there exist integers $n_1, n_2, \ldots, n_n$ such that $n_1e_1+n_2e_2+\ldots+n_ne_n=1$.

Put $\Gamma=\{i\,|\,n_i<0\}$, then $\sum_{i\notin\Gamma} n_ie_i=1+\sum_{i\in\Gamma}(-n_i)e_i$.

Let $P=\sum_{i\in\Gamma}(-n_i)e_i$. then p, $p+1\in S$.

It follows that $\{n\in\mathbf{N}\,|\,n\geqslant e_1p\}\subset S$.

Let (A, m) be a d-dimensional local ring, M be a finitely generated A-module and q be an m-primary ideal. Then the Hilbert polynomial of M is

$$\chi_M{}^q(n)=\frac{e}{d!}n^d+\text{ (lower terms)},\quad e\in\mathbf{N}.$$

This e is denoted by $e(q, M)$ and called the Multiplicity of q. [15, p. 129]

THEOREM 3.2. *Let $A=k[[x^{e_1}, x^{e_2}, \ldots x^{e_n}]]$, $0<e_1<e_2<\ldots<e_n$, $(e_1, e_2, \ldots, e_n)=1$ Then $e(A)=e(m, A)=e_1$ where m is the maximal ideal of A.*

Proof. The maximal ideal of A is $m=(x^{e_1}, x^{e_2}, \ldots, x^{e_n})$.

Let $S_p=\{\sum_{i=1}^{n}a_ie_i\,|\,a_i\geq 0,\ \sum_{i=1}^{n}a_i\geqslant p\}$. Then $S_{p+1}=\bigcup_{i=1}^{n}(S_p+e_i)$.

So $S_{p+1}\supset S_p+e_1$.

If λ_p be the number of elements of the set $(\mathbf{N}+e_1p)-S_p$ then $\{\lambda_p\}$ is a decreasing sequence in the non-negative integers. So it must be stationary. There exists $n>0$ such that $\lambda_n=\lambda_{n+1}$ Consequently $S_{n+1}=$

S_n+e_1.
So $x^{e_1}m^n=m^{n+1}$. This means that (x^{e_1}) is a reduction ideal of m. So

$e(m)=e((x^{e_1}))$. [7, p. 146]

Since A is a Cohen–Macaulay ring,

$e((x^{e_1}))=l(A/x^{e_1})$ where $l(A/x^{e_1})$ is the length of A/x^{e_1}.

Let S be the semi-group generated by $\{e_1, e_2, \ldots, e_n\}$. Then $l(A/(x^{e_1}))$ is the number of elements of the set $S-(S+e_1)$.

Now we claim that the number of elements of $S-(S+e_1)$ is e_1.
Define a mapping γ from $\{0, 1, \ldots, e_1-1\}$ to $S-(S+e_1)$ by

$$\gamma(i)=\min \{x\in S \mid x\equiv i \bmod e_1\}$$

for any $i\in \{0, 1, \ldots, e_1-1\}$.

For p in [Lemma 3–1], assume $e_1(p-j)+i\in S$ and $e_1(p-j-1)+i\notin S$ for $0\leqslant j<p$, then $\{x\in S \mid x-e_1\notin S,\ x\equiv i \pmod{e_1}\}\neq\phi$.
For any $x\in S-(S+e_1)$: $x\in S, x-e_1\notin S,\ x\equiv i \pmod{e_1}$ for $0\leqslant i\leqslant e_1-1$.
Assume $x\neq\gamma(i)$, then $x>\gamma(i)$. So $x-\gamma(i)=pe_1 (p>0)$.

$$x-e_1=\gamma(i)+(p-1)e_1\in S.$$

This is the contradiction to $x-e_1\notin S$. So $x=\gamma(i)$.
Consequently, γ is one-to-one correspondence.
Hence the number of elements of the set $S-(S+e_1)$ is e_1.

By [Lemma 3–1], S has a tail, that is, there is an element $t=\sup(\mathbf{N}-S)$. S is called symmetric if S satisfies the property:

$$s\in S \text{ iff } t-s\notin S.$$

PROPOSITION 3–3. *The subring* $k[[x^{e_1}, x^{e_2}, \ldots, x^{e_n}]]$ *is Gorenstein iff the semi-group* S *is symmetric.* [3, p. 749]

4. Dualizing Complex for $k[[x^{e_1}, x^{e_2}, \ldots, x^{e_n}]]$

Let (A, m, k) be a local ring. If an A-module F is an essential extension of k over A and E is the injective envelope of F over A, then E is also the injective envelope of k over A.

Let $B_i=\{\xi\in F \mid m^i\cdot\xi=0\}$. If $l(B_i)=l(A/m^i)$ for all i, then $F=E$. This follows from [5, Th 3–10, p. 524].

THEOREM 4. 1. *Let* $A=k[[x^{e_1}, x^{e_2}, \ldots x^{e_n}]]\subset R=k[[x]]$. *Let* S *be the* *semi-group generated by* $\{e_1, e_2 \ldots, e_n\}$, *and* $\mathbf{N}-S=\{\beta_1, \beta_2, \ldots, \beta_\lambda\}$. *Set* $N=\sum_{i=1}^{\lambda} kf_{\beta_i}$ *in the injective envelope of* $E_R(k)$ *of* k *over* R. *Then* $F=E_R(k)/N$ *is an injective envelope of* k *over* A.

Proof. $E_R(k)=k[x^{-1}]=K/R=kf_0+kf_1+\ldots$ where $f_i\equiv 1/x^{i+1} \bmod R$. $F=E_R(k)/N=\sum_{i\in S} k\bar{f}_i$. For any $x^{e_i}(i=1, 2, 3, \ldots, n)$ and $f_{\beta_j}(j=1, 2, \ldots, \lambda)$

$$f_{\beta_j}\cdot x^{e_i}=f_{\beta_j-e_i}.$$

Suppose $\beta_j-e_i\notin\mathbf{Z}-S$ for some i, j then $\beta_j-e_i\in S$. So $\beta_j\in S$, a contradiction. So $\beta_j-e_i\in\mathbf{Z}-S$. This implies that N is an A-module. So F is a A-module.

Now k is a submodule of F since k is isomorphic to $k\bar{f}_0$.
For any non-zero element $\xi\in F$

$$\xi=a_0\bar{f}_0+\ldots\ldots+a_n\bar{f}_n$$

where $a_i\in A$, $n\notin\{\beta_1, \beta_2, \ldots, \beta_\lambda\}$.
So x^n is an element of A and satisfies $x^n\xi=a_n\bar{f}_0\in k\bar{f}_0$, $x^n\xi\neq 0$.
If follows that F is an essential extension on k over A.

Next we claim that

$$l(B_n)=l(A/m^n) \text{ for all } n>0$$

where $B_n=\{\xi\in F\,|\,m^n\cdot\xi=0\}$.

Let $S_p=\{\sum_{i=1}^{n} a_ie_i\,|\,a_i\geqslant 0, \sum a_i\geqslant p\}$ as in [Th. 3-2]. Then

$$A/m^n=\sum_{j\in S-S_n} kx^j.$$

And $B_n=\sum_{i\in S-S_n} k\bar{f}_i$ by some calculation. That is $l(A/m^n)=l(B_n)$. So F is the injective envelope of k over A.

THEOREM 4.2. *Assumption is the same as in* [*Th.* 4-1] *and let* $K=Q(A)$. *Then*

$$I^{\cdot}: 0\longrightarrow K\longrightarrow E_A(k)\longrightarrow 0$$

is a Dualizing Complex for A.

Proof. From [Th. 4-1], we get an exact sequence:

$$0\longrightarrow N\longrightarrow E_R(k)\longrightarrow E_A(k)\longrightarrow 0.$$

Now we have the following commutative diagram:

$$\begin{array}{cccccccc}
 & & & & & & 0 & \\
 & & & & & & \downarrow & \\
 & & & & \phi^{-1}(N) & \longrightarrow & N & \\
 & & & & \downarrow & & \downarrow & \\
 & 0 \longrightarrow & R & \longrightarrow & K & \xrightarrow{\phi} & E_R(k) & \longrightarrow 0 \\
 & & & & \| & \searrow & \downarrow g & \\
I^{\cdot}: & & 0 & \longrightarrow & K & \xrightarrow{\partial} & E_A(k) & \longrightarrow 0 \\
 & & & & & & \downarrow & \\
 & & & & & & 0 &
\end{array}$$

where ∂ is defined by $g\circ\phi$. Then $I^{\cdot}$ is a bounded injective complex and $H^1(I^{\cdot})=0$, $H^0(I^{\cdot})\cong\phi^{-1}(N)=R+\sum_{i=1}^{l} kf_{\beta_i}$ hence $I^{\cdot}$ has finitely generated cohomology.

It remains to prove that

$$\alpha^{\cdot}:\ A\longrightarrow \mathrm{Hom}_A(I^{\cdot}, I^{\cdot})$$

is a quasi-isomorphism.

In the same way in [12, p. 375], we consider the following diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow \alpha_0 & & \downarrow \\
0 \longrightarrow & & \begin{matrix}\mathrm{Hom}_A(E,E)\\ \oplus \\ \mathrm{Hom}_A(K,K)\end{matrix} & \xrightarrow{\partial_{F^{\cdot}}} & \mathrm{Hom}_A(K,E) \longrightarrow 0
\end{array}$$

where $\alpha_0(a)=(a,a)$ for any $a\in A$, and

$\partial_{F^{\cdot}}(\phi,\psi)=\phi\cdot\partial+(-1)^1\partial\cdot\psi$ for any $\phi\in\mathrm{Hom}_A(E,E)$, $\psi\in\mathrm{Hom}_A(K,K)$.

Here, $\mathrm{Hom}_A(E,E)\cong A$ [5, p. 522]
$\mathrm{Hom}_A(K,K)\cong K$
$\mathrm{Hom}_A(K,E)\cong K$. [6, p. 575]

So, $\ker(\partial_{F^{\cdot}})=\{(a,a)\in A\oplus K\mid a\in A\}\cong A$

$\partial_{F^{\cdot}}(a,b)=a-b$ for any $a\in A, b\in K$.

So $\partial_{F^{\cdot}}$ is surjective. It follows that $\alpha^{\cdot}$ is a quasi-isomorphism.

THEOREM 4.3. *Let the assumption be the same [Th. 4-1]. Then the Local Cohomology* $H_m{}^1(A)\cong K/A$ *satisfies* $H_m{}^1(A)\cong D(H^0(I^{\cdot}))$, *and* $H^0(I^{\cdot})\cong R+N'$ *where* $N'=\sum_{i\in N-S} k\cdot 1/x^{i+1}$. *If A is Gorenstein, then*

$H^0(I^\cdot)\cong A$.

Proof. By [Proposition 1. 1], $t=t(m;I^\cdot)=1$.

$$H_m{}^1(A)\cong D(H^0(I^\cdot))$$

where $H^0(I^\cdot)=\ker\partial=R+N'$.

If A is Gorenstein, then by [Th. 2-1],

$$H^0(I^\cdot)\cong D(H_m{}^1(A))=\mathrm{Hom}_A(E,E)\cong A.$$

5. Examples

Consider the subrings $A=k[[x^4,x^5]]$, $B=k[[x^3,x^5,x^7]]$ of $R=k[[x]]$.

By using [Proposition 3-3],

A is Gorenstein for $S_A=\{0,(),(),(),4,5,(),(),8,9,10,(),12,13,\ldots\}$ is symmetric. Take $N=kf_1+kf_2+kf_3+kf_6+kf_7+kf_{11}$, a A-submodule of the injective envelope of k over R. Then $H_m{}^1(A)\cong E_A(k)=E_R(k)/N$ $=k\bar{f}_0+k\bar{f}_4+k\bar{f}_5+k\bar{f}_8+k\bar{f}_9+k\bar{f}_{10}+k\bar{f}_{12}+\ldots$

and $H^0(I^\cdot)\cong R+N'$ [Th. 4-3]

$$=R+kx^{-2}+kx^{-3}+kx^{-4}+kx^{-7}+kx^{-8}+kx^{-12}.$$
$$\cong k+kx^4+kx^5+kx^8+kx^9+kx^{10}+x^{12}R.$$
$$\cong A.$$

B is not Gorenstein for $S_B=\{0,(),(),3,(),5,6,7,\ldots\}$ is not symmetric. Take $N=kf_1+kf_2+kf_4$, a B-submodule of the injective envelope of k over R. Then $H_m{}^1(B)\cong K/B\cong D(H^0(I^\cdot))$, where $H^0(I^\cdot)\cong R+N'=R$ $+kx^{-2}+kx^{-3}+kx^{-5}$.

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