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ON ANALYTIC SPECTRAL RESOLVENTS

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Introduction A spectral resolvent E maps an open set in the complex plane to an invariant subspace of a linear operator T in a Banach space. We consider E whose range is contained in the class of all analytically invariant subspaces of T and we call E an analytic spectral resolvent. Obviously an analytic spectral resolvent is a spectral resolvent but the converse is not true in general. In this paper we will show that if T has two-analytic spectral resolvent then the dual operator T^* has also a two-analytic spectral resolvent and we will give some reasonable dualities for an operator with an analytic spectral resolvent.

Throughout this paper, T is an element of $B(X)$, the Banach algebra of bounded linear operators acting on the Complex Banach space X. Let $\sigma(T)$ denote the spectrum of T, $\rho(T)$ the resolvent set of $T, R_\lambda(T)$ the resolvent operator and $\sigma(x, T)$ the local spectrum of T at $x \in X$. We write X^* for the dual space of X. For a set S, S^c is the complement of $S, \bar{S}$ is the closure of S. $\mathrm{Cov}(S)$ stands for the family of all finite open covers of S, $Y^\perp$ is the annihilator of Y in Y^*. $\mathscr{U}$ denotes the collection of all open subsets of $\mathbf{C}$. And $\mathrm{Inv}(T)$, Ana. $\mathrm{inv}(T)$ denote the class of all closed invariant subspaces, and analytically invariant subspaces of T respectively.

If T has the single valued extension property (SVEP), then we write

$$X_T(S) = \{x \in X : \sigma(x, T) \subset S\} \text{ for } S \subset \mathbf{C}.$$

1. DEFINITION. An invariant subspace Y of T is called analytically invariant if, for each X-valued analytic function f defined on a region $D \subset \mathbf{C}$ such that $(\lambda - T)f(\lambda) \in Y$ for $\lambda \in D$, $f(\lambda) \in Y$ for $\lambda \in D$.

For $Y \in Ana.$ $\mathrm{inv}(T)$, the following hold:

$$\sigma(T|Y) \subset \sigma(T), \quad \sigma(T) = \sigma(T|Y) \cup \sigma(T/Y)$$

([8], Corollary 1.4, Proposition 1.7), where $T|Y$, T/Y are the re-

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striction and coinduced operator on the quotient space X/Y respectively.

2. DEFINITION. E is said to be analytic spectral resolvent of T if

(i) $E:\mathfrak{U}\to Ana.\ \mathrm{inv}(T)$

(ii) $E(\phi)=\{0\}$

(iii) for $\{G_i\}^n{}_{i=1}\in\mathrm{Cov}[\sigma(T)]$, $X=\sum_{i=1}^{n}E(G_i)$ $(n\in N)$, and

(iv) $\sigma(T|E(G))\subset\overline{G}$ for each $G\in\mathfrak{U}$.

When $n=2$, we call E a two-analytic spectral resolvent for T. In this definition, if we replace Ana. inv(T) by Inv(T) then E is a spectral resolvent for T.

For example, let T be decomposable. Define $E(G)=X_T(\overline{G})$ $(G\in\mathfrak{U})$, then $X_T(\overline{G})$ is spectral maximal, thus analytically invariant under T. And for every $\{G_i\}\in\mathrm{Cov}[\sigma(T)]$, $\{(G_i, X_T(\overline{G}_i)\}$ is a spectral decomposition of X by T. Therefore $E:\mathfrak{U}\to Ana.\ \mathrm{inv}(T)$ is an analytic spectral resolvent for T.

3. THEOREM. *Let T have an analytic spectral resolvent E. Then T has the SVEP.*

Proof. Let $f:D\to X$ be analytic for every component of an open set $D\subset\mathbf{C}$ verifying the identify

$$(\lambda-T)f(\lambda)=0 \text{ for any } \lambda\in D.$$

Without loss of generality, we may assume that D is connected. Suppose that f is nonzero on D, then $D\subset\sigma(T|E(G))\subset\overline{G}$ for any $G\in\mathfrak{U}$. This follows from Theorem 1. 8, [8]. Hence

$$D\subset\cap\{\overline{G}:G\in\mathfrak{U}\}=\phi,$$

which is a contradiction, thus $f=0$ on D.

It is known that if T has a spectral resolvent then T is decomposable ([10], Theorem 11). In the proof of this theorem, we see that if, T has a two-spectral resolvent then T is decomposable. Just the same calculation as this or by [5], Lemma 17 we have the following theorem.

4. THEOREM. *Let T have a two analytic spectral resolvent, then T*

is decomposable.

The Hahn-Banach theorem leads to the following Lemma.

5. LEMMA. *If M is a linear subspace of a normed linear space X, then every continuous linear form g on M may be extended to a continuous linear form f on X such that $\|f\|=\|g\|$.*

6. THEOREM. *Let T have a two-analytic spectral resolvent E. Then T^* has also a two-analytic spectral resolvent E^*.*

Proof. Let us define $E^* : \mathcal{U}\to X^*$ by $E^*(G)=E(\mathbf{C}\backslash\overline{G})^{\perp}$ for any $G\in \mathbf{C}$, and we prove that E^* is an two-analytic spectral resolvent of T^*.

Since $E(G)=X$ provided $\sigma(T)\subset G, E^*(\phi)=E(\mathbf{C})^{\perp}=X^{\perp}=\{0\}$. And $E^*(G)$ is invariant under T^* for each $G\in\mathcal{U}$; for, if

$$x\in E(\overline{G}^c),\ f\in E(\overline{G}^c)^{\perp}$$

then

$$0=f(Tx)=(T^*f)x,$$

thus $T^*f\in E^*(G)$, that is, $T^*E^*(G)\subset E^*(G)$.

From the identification $[X/E(G)]^*\equiv E(G)^{\perp}$ for any G, we have

$$\sigma(T^*|E^*(G))=\sigma\{[T/E(\overline{G}^c)]^*\}=\sigma(T/E(\overline{G}^c))\subset\overline{G}, \tag{6.1}$$

the last inclusion holds since $E(\overline{G}^c)$ is analytically invariant under T ([5], p. 60, Theorem 10).

Next, we claim that $X^*=E^*(G_1)+E^*(G_2)$ for any $\{G_1, G_2\}\in\mathrm{Cov}\,\sigma(T^*)$. Since $\sigma(T)=\sigma(T^*)$, $\sigma(T)\subset G_1\cup G_2$ so we have $G_1{}^c\cap G_2{}^c\subset\rho(T)$. Hence

$$X_T(G_1^c\cap G_2^c)=X_T(G_1^c\cap G_2^c\cap\sigma(T))=X_T(\phi)=\{0\}.$$
$$\therefore\ X_T(G_1^c)\cap X_T(G_2^c)=\{0\}.$$

A simple computation shows that $E(G)\subseteq X_T(\overline{G})$ for any $G\in\mathcal{U}$, so we have $E(\overline{G}^c)\subset X_T(\overline{G}^c)=X_T(\overline{G}^c)$. Therefore, $E(\overline{G}_1^c)\cap E(\overline{G}_2^c)=\{0\}$ and $E(G_1^c)+E(G_2^c)$ is a direct sum and

$$E(\overline{G}_1^c)\oplus E(\overline{G}_2^c)\subset X_T(G_1^c\cup G_2^c\subset)X.$$

Hence $E(\overline{G}_1^c)\oplus E(\overline{G}_2^c)=Y$ can be considered as a closed linear subspace of X.

Now, let $f\in X^*$ be arbitrary and define $\tilde{g}$ by

$$\tilde{g}(x)=f(x_2), \tag{6.2}$$

where $x\in Y$ and x_2 is the component of x in $E(\overline{G}_2^c)$. Then $\tilde{g}$ is the well defined continuous linear form on Y, by Lemma 5, $\tilde{g}$ can be extended to the continuous linear form $g\in X^*$ such that $\|\tilde{g}\|=\|g\|$.

For any $x\in E(\overline{G}_1^c)$, by (6.2),

$$g(x)=\tilde{g}(x)=f(0)=0.$$

It follows that $g\in E(\overline{G}_1^c)^{\perp}$.

We put $h=f-g$ or $f=g+h$ $(h\in X^*)$. For any $y\in E(\overline{G}_2^c)$,

$$h(y)=f(y)-g(y)=f(y)-\tilde{g}(y)=f(y)-f(y)=0 \text{ by } (6.2),$$

whence $h\in E(\overline{G}_2^c)^{\perp}=E^*(G_2)$, and we have

$$f=g+h\in E^*(G_1)+E^*(G_2) \text{ for any } f\in X^*.$$
$$\text{i.e. } X^*=E^*(G_1)+E^*(G_2).$$

It remains to prove that $E^*(G)$ is analytically invariant under T^* for each $G\in\mathfrak{U}$: Let $f^*: D\to X^*$ be analytic on an open set $\phi\neq D\subset\mathbf{C}$ such that

$$(\lambda-T^*)f^*(\lambda)\in E^*(G) \text{ for any } \lambda\in D.$$

Without loss of generality we may assume that D is connected. We denote $\langle x, f\rangle=f(x)$ in the sequel.

(a) Assume $D\subset\overline{G}$.

Since $\sigma[T|E(\overline{G}^c)]\subset G^c$ and D is open,

$$D\subset G\subset\rho[T|E(\overline{G}^c)].$$

For any $x\in E(\overline{G}^c)$ and any $\lambda\in D$, $R_\lambda(T|E(\overline{G}^c))x$ is defined and we have

$$\langle x, f^*(\lambda)\rangle=\langle(\lambda-T)R_\lambda(T|E(\overline{G}^c))x,\ f^*(\lambda)\rangle$$
$$=\langle R_\lambda(T|E(\overline{G}^c))x,\ (\lambda-T^*)f^*(\lambda)\rangle=0$$

since $R_\lambda(T|E(\overline{G}^c))x\in E(\overline{G}^c)$ and $(\lambda-T)^*f^*(\lambda)\in E^*(G)$.

Therefore $f^*(\lambda)\in E(\overline{G}^c)^{\perp}=E^*(G)$ for any $\lambda\in D$.

(b) Assume $D\not\subset\overline{G}$.

Since $\sigma[T^*|E^*(G)]\subset\overline{G}\,(G\in\mathfrak{U})$, for any $\lambda\in D\backslash\overline{G}$ the resolvent operator $R_\lambda(T^*|E^*(G))$ is defined and

$$(\lambda-T^*)R_\lambda(T^*|E^*(G))=I^*|E^*(G).$$

Thus

$$(\lambda - T^*)\{f^*(\lambda) - R_\lambda[T^* | E^*(G)](\lambda - T^*)f^*(\lambda)\} = 0.$$

Since T is decomposable, so is T^* ([7]), whence T^* has the SVEP. It follows that

$$f^*(\lambda) = R_\lambda[T^* | E^*(G)](\lambda - T^*)f^*(\lambda) \in E^*(G)$$

for any $\lambda \in D \backslash \overline{G}$. Hence $f^*(\lambda) \in E^*(G)$ for any $\lambda \in D$ by analytic continuation. This completes the proof.

For each $G \in \mathfrak{U}$, $E(G)$ is analytically invariant under T, thus we have

$$(6.3) \qquad \sigma(T | E(G)) \subset \overline{G} \cap \sigma(T), \quad \sigma(T/E(G)) \in G^c \cap \sigma(T)$$

and

$$\sigma(T | E(G)) \cup \sigma(T/E(G)) = \sigma(T).$$

For the duality we have a following

7. COROLLARY. *Let T have an analytic resolvent E. Then*

$$\sigma(T^* | E^*(G)) \subset \overline{G} \cap \sigma(T^*), \quad \sigma(T^*/E^*(G)) \subset G^c \cap \sigma(T^*)$$

and

$$\sigma(T^* | E^*(G)) \cup \sigma(T^*/E^*(G)) = \sigma(T^*), \quad (G \in \mathfrak{U}).$$

8. COROLLARY. *Let X be the reflexive Banach space. Then with the same definition of E^* as in Theorem 6, $E^{**} = (E^*)^*$ is the two-analytic spectral resolvent of T^{**} and $E = E^{**}$ on $\mathfrak{U}$.*

Proof. Put $F(G) = E^*(G) = [E(\overline{G}^c)]^\perp$, $G \in \mathfrak{U}$. Then $F^* : \mathfrak{U} \to X^{**}$ defined by $F^*(G) = F(\overline{G}^c)^\perp$ is the two-analytic spectral resolvent of $T^{**} (\equiv T)$ by Theorem 6. And

$$F^*(G) = E(G)^{\perp\perp} = E(G)$$
i. e. $E^{**}(G) = E(G)$ for $G \in \mathfrak{U}$.

The duality for the inclusion $E(G) \subset X_T(\overline{G})$ is also valid: $E^*(G) \in X^*_{T^*}(\overline{G})$, $G \in \mathfrak{U}$.

For, let $x^* \in E^*(G)$ then, since $E^*(G) \in$ Ana. inv (T^*)

$$\sigma(x^*, T^*) = \sigma[x^* | (T^* | E^*(G))] \subset \sigma(T^* | E^*(G)) \subset \overline{G}.$$

Thus $\qquad x^* \in X^*_{T^*}(\overline{G})$ for any $x^* \in E^*(G)$.

We now discuss conditions for which the 'union $\sigma(T|E(G)) \cup \sigma(T/E(G)) = \sigma(T)$ is disjoint. For this purpose we use the following definition: For an operator T with disconnected spectrum, if there is an open set G with the following properties;

$$\sigma(T) \not\subset G, G \cap \sigma(T) \neq \phi \text{ and } \partial G \subset \rho(T),$$

where ∂G is the boundary of G, then we says that G disconnects the spectrum $\sigma(T)$.

9. PROPOSITION. *Let E be an analytic spectral resolvent for T. If $G \in \mathfrak{U}$ disconnects the spectrum $\sigma(T)$, then both $\sigma(T|E(G))$ and $\sigma(T|E(G))$ and $\sigma(T/E(G))$ are separate parts of $\sigma(T)$.*

Proof. It is enough to show that

$$\sigma(T|E(G)) = \overline{G} \cap \sigma(T), \quad \sigma(T/E(G)) = G^c \cap \sigma(T).$$

These follow from the followings;

$$\sigma(T) = \sigma(T|E(G)) \cup \sigma(T/E(G)) \subseteq [\overline{G} \cap \sigma(T)] \cup [G^c \cap \sigma(T)] = \sigma(T),$$
$$\text{i.e. } \sigma(T) = [\overline{G} \cap \sigma(T)] \cup [G^c \cap \sigma(T)] = \sigma(T|E(G)) \cup \sigma(T/E(G)).$$

And $\quad \sigma(T|E(G)) \cap \sigma(T/E(G)) \subset \sigma(T) \cap \overline{G} \cap G^c = \sigma(T) \cap \partial G = \phi.$

It follows that $\sigma(T|E(G)) = \overline{G} \cap \sigma(T)$, $\sigma(T/E(G)) = G^c \cap \sigma(T)$, and $\sigma(T)$ is the disjoint union of these two sets.

10. COROLLARY. *Under the same notation as in the Theorem 6, we have*

$$\sigma(T^*|E^*(G)) = \overline{G} \cap \sigma(T^*),$$
$$\sigma(T^*/E^*(G)) = G^c \cap \sigma(T^*),$$

where $G \in \mathfrak{U}$ disconnects the spectrum $\sigma(T^*) = \sigma(T)$.

11. COROLLARY. *Under the same conditions as the proposition 9, there are bounded open sets G_1 and G_2 such that*

$$E(G_1) \oplus E(G_2) = X,$$
$$\sigma(T|E(G_1)) \cup \sigma(T|E(G_2) = \sigma(T), \text{ the union is disjoint.}$$

Proof. By the proposition 9, both $\sigma(T|E(G))$ and $\sigma(T/E(G))$ are bounded closed disjoint subsets of $\sigma(T)$ provided G disconnects $\sigma(T)$. Thus there are bounded disjoint open sets G_1, G_2 in $\mathbf{C}$ such that

$$\sigma(T|E(G))\subset G_1,\ \sigma(T/E(G))\subset G_2 \text{ and}$$
$$\partial G_i\subset\rho(T)\quad(i=1,2).$$

Thus

$$\{G_1, G_i\}\in\text{Cov }\sigma(T)\ \text{ and }\ E(G_1)+E(G_2)=X.$$

The condidition $\partial G_i\subset\rho(T)$ implies that

$$E(G_i)=X_T(G_i)=X_T(\overline{G}_i)\quad([10],\ \text{Proposition 15}).$$

Therefore

$$E(G_1)\cap E(G_2)=X_T(G_1\cap G_2)=X_T(\phi)=\{0\},$$
$$E(G_1)\oplus E(G_2)=X.$$

Consequently $\sigma(T)=\sigma(T|E(G_1))\cup\sigma(T|E(G_2))$. We see that the union is disjoint, this follows
from the fact the $\sigma(T|E(G_i))\subset\overline{G}_i\cap\sigma(T)=G_i\cap\sigma(T)$.

12. LEMMA. (i) [[8], *Theorem* 1.2] *Let Y be analytically invariant under T, and suppose that for each $x\in X$, we have*

$$\sigma(\hat{X}, T/Y)=\overline{\sigma(x, T)\backslash\sigma(T|Y)},$$

where $\hat{x}$ is the coset of x in X/Y. Then Y is a spectral maximal space of T.

(ii) [[6], *Theorem* 3] *The following assertions are equivalent*:

(1) *T is strongly decomposable*;
(2) (a) *T has the SVEP and $X_T(F)$ is closed for every closed $F\subset\mathbf{C}$,*
(b) *for every spectral maximal space Y of T and any $x\in X$,*
$$\sigma(\hat{x}, T/Y)=\overline{\sigma(x, T)\backslash\sigma(T|Y)},\quad \hat{x}=x+Y,$$
(c) *for every spectral maximal space Y of T and any open $G\subset\mathbf{C}$,*
$G\cap\sigma(T|Y)\neq\phi$ implies that $X_T[\overline{G}\cap\sigma(T|Y)]\neq\{0\}$.

(iii) [[1], (1.1)] *Let Y and Z be invariant under T, if $Y\subset Z$ then*

$$(T|Z)|Y=T|Y$$
$$(T|Z)/Y=(T/Y)|Z/Y.$$

From Lemma 12, (i) and (ii), we have the following

13. COROLLARY. *Let $E:\mathcal{U}\to Ana.\ inv(T)$ be an analytic spectral resolvent for T. Suppose that for every analytically invariant space Y*

of T *the conditions*

$$(*)\ \ \sigma(\hat{x}, T/Y)=\overline{\sigma(x, T)\backslash\sigma(T|Y)},$$

and for any $G\subset\mathbf{C}$, $G\cap\sigma(T|Y)\neq\phi$ *implies that*

$$X_T[\overline{G}\cap\sigma(T|Y)]\neq\{0\},$$

then $T|(G)$ *is decomposable for* $G\in\mathfrak{U}$.

Proof. Since T is decomposable, T has the SVEP and $X_T(F)$ is closed for any closed $F\subset\mathbf{C}$. Every spectral maximal space of T is analytically invariant under T. Therefore, by Lemma 12, (ii), T is strongly decomposable. And for any $G\in\mathfrak{U}$ verifies the condition (*), $E(G)$ is spectral maximal by the Lemma 12, (i). Thus $T|E(G)$ is decomposable.

We notice that a strongly decomposable operator is decomposable, but the converse remains open.

14. THEOREM. *Let* $E:\mathfrak{U}\rightarrow Ana.\ inv(T)$ *be an analytic spectral resolvent for* T, *let* $T|E(G)$ *be decomposable for each* $G\in\mathfrak{U}$. *put* $\overline{TE(G)}=Y(G)$, $\tilde{Y}(G)=E(G)/Y(G)$ *and* $\tilde{T}=T/Y(G)$.

Then $\tilde{Y}(G)$ *is analytically invariant under* $\tilde{T}$; *if* $Y(G)$ *satisfies the condition* (*), *then*

$$\sigma(\tilde{T}|\tilde{Y}(G))\subset\overline{G}\cap\sigma(\tilde{T}). \tag{14.1}$$

Proof. First, we claim that

$$\sigma[\{T|E(G)\}/Y(G)]\subset\overline{G}\cap\sigma(T/Y(G)).$$

Observe that since $Y(G)=\overline{TE(G)}$ is spectral maximal of T, $Y(G)$ is also spectral maximal of $T|E(G)$, this follows from [2] Proposition 3.2, (1). And since $T|E(G)$ is decomposable, we have

$$\begin{aligned}\sigma[\{T|E(G)\}/Y(G)]&=\overline{\sigma(T|E(G))\backslash\sigma[\{T|E(G)\}|Y(G)]}\\&=\overline{\sigma(T|E(G))\backslash\sigma(T|Y(G))}.\end{aligned} \tag{14.2}$$

$E(G)$ is analytically invariant under T implies that $Y(G)$ is analytically invariant under T [8], Proposition 1.17, hence

$$\sigma(T)=\sigma(T|Y(G))\cup\sigma(T/Y(G)).$$

Thus, by (14.2), we have

$$\sigma[\{T|E(G))\}/Y(G)]\subset\overline{\bar{G}\cap\sigma(T)\backslash[\sigma(T)\backslash\sigma(T/Y(G)]}$$
$$=\bar{G}\cap\sigma(T/Y(G)).$$

In general, $\{T|E(G)\}/Y(G)=T/Y(G)|E(G)/Y(G)$ holds by Lemma 12, (iii). Hence we have

$$\sigma[T/Y(G)|E(G)/Y(G)]\subset\bar{G}\cup\sigma(T/Y(G))$$
$$\therefore\ \sigma(\tilde{T}|\tilde{Y}(G))\subset\bar{G}\cup\sigma(\tilde{T}).$$

$E(G)$ is analytically invariant under T implies that $\tilde{Y}(G)$ is analytically invariant under $\tilde{T}=T/E(G)$, this follows from [8], Proposition 1.13.

Without the assumptions that $T|E(G)$ is decomposable and $Y(G)$ is spectral maximal, the dual form of (14.1) is following:

15. THEOREM. *Let $E:\mathfrak{U}\to Ana.\ inv\ (T)$ be an analytic spectral resolvent for T. Under the same notations as in Theorem* 14,

$$\sigma(\tilde{T}^*|\tilde{Y}(G)^{\perp})\subset G^c\cap\sigma(\tilde{T})\quad(G\in\mathfrak{U}),$$

where $\tilde{T}^*\in B([\hat{X}(G)]^*)$, $[\hat{X}(G)]^*$ the dual space of $\hat{X}(G)=X/Y(G)$.

Proof. Since $\sigma(T^*|E(G)^{\perp})=(T/E(G))$ and $[T/Y(G)]^*|\tilde{Y}(G)^{\perp}$ is unitarily equivalent to an operator $T^*|E(G)^{\perp}$, this follows from [6], Lemma 5, we have

$$\sigma(\tilde{T}^*|\tilde{Y}(G)^{\perp})=\sigma(T^*|E(G)^{\perp})=\sigma(T/E(G)).$$

Therefore we have

$$\sigma(\tilde{T}^*|\tilde{Y}(G)^{\perp}\subset G^c\cap\sigma(T).\tag{15.1}$$

On the other hand $\tilde{Y}(G)$ being analytically invariant under $\tilde{T}$,

$$\sigma(\tilde{T}|\tilde{Y}(G))\cup\sigma(\tilde{T}/\tilde{Y}(G))=\sigma(\tilde{T}).$$

Put $\hat{X}(G)=X/Y(G)$, then $\tilde{Y}(G)$ is a closed linear subspace of $\hat{X}(G)$ for the quotient norm. Moreover, since $\tilde{T}\in B(\hat{X}(G))$ and $\tilde{Y}(G)^{\perp}\equiv[\hat{X}(G)/\tilde{Y}(G)]^*$ we have

$$\sigma(\tilde{T}^*|\tilde{Y}(G)^{\perp})=\sigma[(\tilde{T}/\tilde{Y}(G))^*]=\sigma(\tilde{T}/\tilde{Y}(G)).\tag{15.2}$$

It follows that

$$\sigma(\tilde{T}^*|\tilde{Y}(G)^{\perp})\subset\sigma(\tilde{T}).\tag{15.3}$$

From (15.1) and (15.3), we have

$$\sigma(\tilde{T}^* \mid \tilde{Y}(G)^{\perp}) \subset G^c \cap \sigma(\tilde{T}).$$

$Y(G) = \overline{TE(G)}$ being analytically invariant under T, obviously $\sigma(\tilde{T}) \subset \sigma(T)$.

By the same calculation as in the proof of proposition 9, we have a following corollary:

16. COROLLARY. *Under the same assumptions and notations as in Theorem 14, if G disconnects the spectrum $\sigma(\tilde{T})$, then both $\sigma(\tilde{T} \mid \tilde{Y}(G))$ and $\sigma(\tilde{T}/\tilde{Y}(G))$ are separate parts of $\sigma(\tilde{T})$; $\sigma(\tilde{T})$ is the disjoint union of these separate parts.*

REMARK. Here we see that $\tilde{Y}$ is not an analytic spectral resolvent for $\tilde{T}$, but it plays similar role as E at least on the properties (6.3) in this note.

If T has a spectral resolvent E and G is open, then

$$\overline{G \cap \sigma(T)} \subset \sigma(T \mid E(G)) \subset \bar{G} \quad \text{([10], Proposition16)}.$$

If E is an analytic spectral resolvent, we obtain the same inclusion relation as the above through much easier calculation: If $E(G)$ is analytic invariant, then

$$\sigma(T \mid E(G)) \supset \sigma(T) \backslash \sigma(T/E(G)) \supset \sigma(T) \backslash G^c \cap \sigma(T) = G \cap \sigma(T),$$

thus

$$\overline{G \cap \sigma(T)} \subset \sigma(T \mid E(G)) \subset \bar{G} \cap \sigma(T).$$

17. COROLLARY. *Under the same assumption as in Theorem 15, we have*

$$\overline{\sigma(\tilde{T}) \cap G} \subset \sigma(\tilde{T} \mid \tilde{Y}(G)).$$

This follows from Theorem 15 and the same computation as the above. And if $Y(G)$ is spectral maximal, $T \mid E(G)$ is decomposable then, by Theorem 14,

$$\overline{\sigma(\tilde{T}) \cap G} \subset \sigma(\tilde{T} \mid \tilde{Y}(G)) \subset \bar{G} \cap \sigma(\tilde{T}).$$

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