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QUANTUM STATISTICAL MECHANICS OF UNBOUNDED CONTINUOUS SPIN SYSTEMS

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1. Introduction

In this paper we develop the quantum statistical mechanics of unbounded d-component continuous spin systems on the lattice space Z^ν interacting via potentials which are superstable and low regular. The models we consider can be viewed as models for quantum anharmonic crystals and are closely connected to lattice field theory with continuous times. Thus the models are of interest in their own right and as well as in connection with the Hamiltonian formalism of quantum field theory in lattice spaces.

In the case of unbounded classical spin systems with superstable interactions there has been extensive studies on the infinite volume limit of the pressure and the infinite volume Gibbs states and fairly satisfactory results such as probability estimates [14, 12], existence of the infinite volume free energy density [1] and equilibrium equations (DLR equations) for the infinite volume limit states [7, 3] have been obtained. In a region of high temperatures analyticity and clustering properties has been established [8]. The Gibbs states for quantum anharmonic oscillators with gentle (bounded) perturbations have been studied in details by Albeverio and Hoegh-Krohn [1]. We mention that the Ruelle's results on superstable interactions in classical statistical mechanics have been partially extended to quantum statistical mechanics [10]. The main purpose in this paper is to extend the results on the unbounded classical spin systems to the quantum cases. The main methods we use are Ruelle's probability estimates [14, 13] and the Wiener integral formalism in statistical mechanics. In the region of high temperatures we use the cluster expansion method developed in [9].

Our main results can be summarized as follows: We first establish

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basic apriori estimates similar to those of Ruelle's probability estimates. Using the estimate we show that the thermodynamic limit of the pressure exists in the sense of van Hove. We next construct infinite volume (physical) Hilbert spaces with a time translation invariant vector and strongly continuous unitary representations of time evolutions. We actually give two construction procedures. In either case it turns out that infinite volume states are modular and satisfy (weak) KMS condition. Finally, in the region of high temperatures there exists a unique Hilbert space with a vector invariant under time and lattice translations. For the details, see Section 2.

We list the contents of this paper. In Section 2 we introduce necessary notations and definitions for the models. We then list assumptions on the interactions (Assumption 2. 1–Assumption 2. 3). In Section 3. 1 we review briefly the Wiener integral formalism in statistical mechanics. Assuming the apriori estimates (Proposition 2. 1) we prove the existence of the infinite limit of the pressure (Theorem 2. 2), and main results on infinite volume theory (Theorem 2. 3–Theorem 2. 4) in Section 3. 2 and Section 3. 3, respectively.

In Section 4 we prove the basic estimates (Proposition 2. 1). The main tools we use are the method similar to that of Ruelle [14] and the Wiener Integral Formalism. Section 5 is devoted to study of the thermodynamic limit theory of the systems in the region of high temperatures. We use the cluster expansion method [9] to show the infinite volume limits of finite volume generating functionals exist and satisfy the cluster property. From these facts we prove Theorem 2. 5.

Admittedly we left several important questions unanswered. Thus we discuss some topics related to infinite volume Hilbert spaces, phase transition and continuous symmetry breaking in Section 6.

2. Notations, assumptions and main results

We consider unbounded quantum spin systems on the lattice space Z^ν. At each lattice site $i\in Z^\nu$ we associated an identical copy of the Hilbert space $L^2(\mathbf{R}^d, dx)$, where dx is the Lebesgue measure on R^d. For $x=(x^1, \dots, x^d)\in R^d$, $i=(i_1, \dots, i_\nu)\in Z^\nu$ we write

$$|x|=\Big(\sum_{i=1}^{d}(x^i)^2\Big)^{1/2}, \quad |i|=\max_{1\le l\le\nu}|i_l|. \tag{2.1}$$

For a bounded region $\Lambda\subset Z^d$ we write

$$X_\Lambda=\{x_i : i\in\Lambda\},\quad dx_\Lambda=\prod_{i\in\Lambda} dx_i. \tag{2.2}$$

The Hilbert space for unbounded quantum spin systems in a finite region $\Lambda\subset Z^\nu$ is given by

$$\begin{aligned}\mathcal{H}_\Lambda&=\bigotimes_{i\in\Lambda} L^2(R^d,\ dx_i)\\&=L^2((R^d)^\Lambda, dx_\Lambda).\end{aligned} \tag{2.3}$$

We introduce a Hamiltonian operator on $\mathcal{H}_\Lambda$ by

$$H_\Lambda=-\frac{1}{2}\sum_{i\in\Lambda}\Delta_i+U(x_\Lambda) \tag{2.4}$$

where Δ_i is the Laplacian for the variables $x_i\in R^d$ and $U(x_\Lambda)$ is the interaction which is a real valued function on $(R^d)^\Lambda$. We assume that U satisfies the following conditions [7, 14]:

ASSUMPTION 2.1. The interaction U is a real function on $(R^d)^\Lambda$ for all finite $\Lambda\subset Z^\nu$ satisfying the following conditions:

(a) $U(x_\Lambda)$ is a Borel function on $(R^d)^\Lambda$

(b) U is invariant under translations of Z^ν

(c) Superstability. There are $A>0$ and $c\in R$ such that for every $x_\Lambda\in(R^d)^\Lambda$

$$U(x_\Lambda)\geq\sum_{i\in\Lambda}[Ax_i^2-c].$$

(d) Regularity. There exists a decreasing positive function ψ on natural integers such that

$$\sum_{i\in Z^\nu}\psi(|i|)<\infty.$$

Furthermore, if Λ_1, Λ_2 are disjoint finite subsets of Z^ν and if one write

$$U(x_{\Lambda_1\cup\Lambda_2})=U(x_{\Lambda_1})+U(x_{\Lambda_2})+W(x_{\Lambda_1}, x_{\Lambda_2}) \tag{2.5}$$

then

$$|W(x_{\Lambda_1}, x_{\Lambda_2})|\leq\sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2}\psi(|i-j|)\frac{1}{2}(x_i^2+x_j^2).$$

(e) There is a bounded Borel set Σ in R^d and $\lambda>0$ such that

$$Tr_{L^2(\Sigma^\Lambda)}(\exp(-H_\Lambda))>\lambda^{|\Lambda|}$$

where $|\Lambda|=\text{card}(\Lambda)$ and the Laplacians in H_Λ are replaced by the Laplacians with Dirichlet boundary conditions on $\partial\Sigma$.

For a bounded domain $\Lambda\subset Z^\nu$ the local C^*-algebra is defined by

$$\mathcal{A}_\Lambda=\mathcal{L}(\mathcal{H}_\Lambda) \tag{2.6}$$

where $\mathcal{L}(\mathcal{H}_\Lambda)$ is the algebra of all bounded operators on $\mathcal{H}$. If $\Lambda_1\cap\Lambda_2=\phi$, then $\mathcal{H}_{\Lambda_1\cup\Lambda_2}=\mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the C^*-algebra of $\mathcal{A}_{\Lambda_1}\otimes 1_{\Lambda_2}$, where 1_{Λ_2} denotes the identity operator on $\mathcal{H}_{\Lambda_2}$. In this way we identify $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}_{\Lambda'}$ if $\Lambda\subset\Lambda'$.
Let

$$\mathcal{A}=\overline{\bigcup_{\substack{\Lambda\subset Z_\nu \\ \Lambda:\text{finite}}}\mathcal{A}_\Lambda} \tag{2.7}$$

be the algebra of the quasi-local observables. Notice that $\mathcal{A}$ has an identity. The finite volume partition function and the finite volume Gibbs state $\omega_{\Lambda,\beta}$ with inverse temperature $\beta>0$ are defined by

$$\begin{aligned} Z_{\Lambda,\beta}&=Tr_{\mathcal{H}_\Lambda}(e^{-\beta\mathcal{H}_\Lambda}) \\ \omega_{\Lambda,\beta}(A)&=Z^{-1}{}_{\Lambda,\beta}Tr_{\mathcal{H}_\Lambda}(Ae^{-\beta\mathcal{H}_\Lambda}),\quad A\in\mathcal{A}_\Lambda. \end{aligned} \tag{2.8}$$

By Assumption 2.1 (c) the operator $\exp(-\beta H_\Lambda)$ belongs to the trace class and so the above definitions make sense.

We now describe main results in this section. For a given bounded subset $B\subset\Lambda$, we write

$$\begin{aligned} \Delta(B)&=\sum_{i\in B}\Delta_i \\ x^2(B)&=\sum_{i\in B}|x_i|^2. \end{aligned} \tag{2.9}$$

We have the following apriori estimates:

PROPOSITION 2.1. *For given B and Λ $(B\subset\Lambda)$ there is a constant $c>0$ independent of B and Λ such that for sufficiently small $\varepsilon<0$ (depended on A in* Assumption 2.1(c))

$$\frac{Tr(\exp[\beta\varepsilon x^2(B)]\exp[-\beta(H_\Lambda+\frac{1}{4}\Delta(B))])}{Tr(\exp[-\beta H_\Lambda])}\leq\exp(c|B|)$$

where $|B|=\text{card}(B)$ *and* $Tr(A)=Tr_{\mathcal{H}_\Lambda}(A)$.

The proof of the above proposition is postponed to Section 4. We define the pressure by

$$P_\Lambda(\beta)=\frac{1}{|\Lambda|}\log Z_{\Lambda,\beta}. \tag{2.10}$$

The following result which we will show in Section 3.2 is a consequence of proposition 2.1.

THEOREM 2.2. *Let $\Lambda\to Z^\nu$ in the sense of van have. Then*

$$P(\beta)=\lim_{\Lambda\to Z^\nu} P_\Lambda(\beta)$$

exists.

We next consider infinite volume equilibrium states for the systems. In the following we drop β in the notations if there are no confusions involved. We define the finite volume Green functions by

$$G_\Lambda(A,B;t)=\omega_\Lambda(A\alpha_t^\Lambda(B)),\quad A,B\in\mathcal{A}_\Lambda \tag{2.11}$$

where α^Λ be the time evolution automorphism on $\mathcal{A}_\Lambda$,

$$\alpha_t^\Lambda(B)=e^{itH_\Lambda}Be^{-itH_\Lambda},\quad B\in\mathcal{A}_\Lambda \tag{2.12}$$

and ω_Λ is the Gibbs state defined in (2.7). Although ω_Λ is defined as a state $\mathcal{A}_\Lambda$, it has an extension to a state on $\mathcal{A}$ by Hahn–Banach theorem. By a slight abuse of language we refer such an extension is ω_Λ (respectively G_Λ). The bounds

$$|G_\Lambda(A,B;t)|\leq\|A\|\|B\| \tag{2.13}$$

imply that there always exists a subnet Λ_α such that

$$G(A,B;t)=\lim_{\Lambda_\alpha\to Z^\nu} G_\Lambda(A,B;t) \tag{2.14}$$

exists for all $A,B\in\mathcal{A}$, $t\in R$[2]. This is a consequence of Tychonoff's theorem. Clearly the values

$$\omega(A)=G(A,\mathbb{1};0) \tag{2.15}$$

determine a state ω over the quasi-local algebra $\mathcal{A}$. Ideally one would like to have an automorphism α on $\mathcal{A}$ (or $\pi(\mathcal{A})''$) such that

$$G(A,B;t)=\omega(A\alpha_t(B)).$$

However, we have a weaker result with an additional assumption on the interactions.

ASSUMPTION 2. 2. *Polynomially boundedness of interactions.* There are a constant $D>0$ and a natural number n such that the one body potentials satisfy the following bound:

$$U(x_i)\leq D(|x_i|^{2n}+1)$$

for any $i\in Z^\nu$.

THEOREM 2. 3. *Let the interaction satisfy* Assumption 2. 1 *and* Assumption 2. 2. *Let* $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ *be the cyclic representation corresponding to the state* ω *defined* (2. 15). *Then there exist a Hilbert space* $\mathcal{H}$ *containing* $\mathcal{H}_\omega$, *and a strongly continuous unitary representation* U *of* R *on* $\mathcal{H}$ *such that*

(a) $\mathcal{H}=\bigvee_{t\in R} U(t)\mathcal{H}_\omega$

(b) $G(A,B;t)=(\pi_\omega(A)^*\Omega_\omega, U(t)\pi_\omega(B)\Omega_\omega)$

(c) [*KMS condition*] *For any* $A, B\in\mathcal{A}$ *and all* f *with* $\hat{f}\in\mathcal{D}$

$$\int_{-\infty}^{\infty} dt\, f(t)\ G(A,B;t)=\int_{-\infty}^{\infty} dt\ f(t+i\beta)\ G(B,A;-t).$$

The properties (a) *and* (b) *determine* $(\mathcal{H}, U)$ *uniquely up to unitary equivalence.*

(d) ω *is a modular state, i.e.,* Ω_ω *is separating for* $\pi_\omega(\mathcal{A})''$.

Let Δ *be the corresponding modular operator associated with the pair* $(\pi_\omega(\mathcal{A})'', \Omega_\omega)$. *Then*

(e) $\|U(i\beta)^{1/2}\phi\|=\|\Delta^{1/2}\phi\|$

for all $\phi\in D(\Delta^{1/2})$.

REMARK. Assumption 2. 2 is needed because of technical reasons in the proof of the above theorem (see the proof of Lemma 3. 3. 1 in Section 3. 3). Refining the method in Section 3. 3 one may be able to prove the above theorem under weak assumptions.

The above construction of the quadruple $(\mathcal{H}, \pi_\omega(\mathcal{A})'', U(t), \Omega_\omega)$ for an infinite volume limit theory is very reasonable in the framework of the C^*-algebraic approach. In the following we give another construction of an infinite volume limit theory, which has a flavour of constructive quantum field theory [6, 16]. In section 3. 1 we will introduce

Wiener measures $P_\Lambda^\beta(x_\Lambda, x_\Lambda; d\omega_\Lambda)$ on $\Omega_\beta{}^\Lambda$, where $\Omega_\beta = C([0, \beta], R^d)$. For $\Lambda \subset Z^\nu$ we define interacting measures on $\Omega_\beta{}^\Lambda$ by

$$dp_{\Lambda,\beta} = Z_{\Lambda,\beta}{}^{-1} e^{-U(\omega_\Lambda)} P_\Lambda{}^\beta(x_\Lambda, x_\Lambda; d\omega_\Lambda) dx_\Lambda \qquad (2.16)$$

where $U(\omega_\Lambda) = \int_0^\beta U(\omega_\Lambda(\tau))d\tau$ and $\omega_\Lambda \in \Omega_\beta{}^\Lambda$. See Section 3.1 for the details. For $f \in C([0, \beta] \times B, \mathbf{C}^d)$, $B \subset \Lambda$, and for $\omega_\beta \in \Omega_\beta{}^\Lambda$ we write

$$\phi(f) = \sum_{i \in B} \int_0^\beta \omega_i(\tau) f(\tau, i) d\tau \qquad (2.17)$$

and

$$J_{\Lambda,\beta}(f) = \int e^{i\phi(f)} dp_{\Lambda,\beta}. \qquad (2.18)$$

The random variables $\phi(f)$ and the characteristic functionals $J_{\Lambda,\beta}(f)$ will play the roles of time-zero fields and generating functionals in quantum field theory. It is easy to check that

(a) $J_{\Lambda,\beta}(0) = 1$
(b) $f \to J_{\Lambda,\beta}(f)$ continuous
(c) $J_{\Lambda,\beta}$ is positive definite
(d) $J_{\Lambda,\beta}(f)$ satisfies reflection positivity on the torus $T_\beta = [0, \beta]$.

In Lemma 3.3 we will show that

(e) there exists a constant c independent of Λ such that

$$|J_{\Lambda,\beta}(f)| \leq \exp(c|||f|||^2) \qquad (2.19)$$

where $|||f||| = \sum_{i \in B} (\int_0^\beta |f(\tau, i)|^2 d\tau)^{1/2}$. Thus there exists a subnet such that

$$J_\beta(f) = \lim_{\Lambda_\alpha \to Z^\nu} J_{\Lambda_\alpha,\beta}(f) \qquad (2.20)$$

exists. The infinite volume generating functionals satisfy (a)-(e) in the above ((b) follows from (e)). We write

$$S_\beta(f_1, \ldots, f_n) = \prod_{i=1}^n \frac{d}{dz_i} J_\beta(z_1 f_1 + \ldots + z_n f_n)|_{z_i=0}. \qquad (2.21)$$

The functions $S_\beta(f_1, \ldots, f_n)$ correspond to the Schwinger functions in field theory [6, 16] and are bounded by $\sqrt{n!}\pi \|f_i\|_2{}^2$ as a consequence of

(2. 19). Hence we can use the same reconstruction process that is used in constructive field theory to have the following result:

THEOREM 2. 4. *Assume the interaction satisfy* Assumption 2. 1.

(a) *There exists a probability measure* dp_β *on* $\Omega_\beta{}^{Z^\nu}$ *such that*

$$J_\beta(f)=\int e^{i\phi(f)}dp_\beta$$

for any $f\in C([0,\beta]\times B,\mathbf{C}^d)$.

(b) *There exists an infinite volume (physical) Hilbert space* $\hat{\mathscr{H}}_\beta$, *a* C^**-algebra* $\mathfrak{U}$, *a cyclic vector* $\hat{\Omega}_\beta$ *and a strongly continuous unitary group* $\hat{U}(t)$ *leaving* $\hat{\Omega}_\beta$ *invariant. Moreover, let*

$$\omega_\beta(A\alpha_t(B))=(\hat{\Omega}_\beta, A\hat{U}(t)B\hat{\Omega}_\beta),\ A,B\in\mathfrak{U}.$$

Then ω_β *is a modular state on* $\mathfrak{U}$ *satisfying the KMS condition.*

REMARK. The C^*–algebra $\mathfrak{U}$ is generated by $\hat{U}(t)\ \exp[i\phi(f)]\hat{U}(t)^*$. $f\in C([0,\beta]\times B, R^d)$, finite $B\subset Z^\nu$, $t\in R$.

We next consider the systems in the region of high temperatures. For technical reasons we assume the following:

ASSUMPTION 2. 3. Let the interaction U be consisted of one and two body potentials:

$$U(x_\Lambda)=\sum_{i\in\Lambda}U(x_i)+\sum_{i,j\in\Lambda}W(x_i,x_j).$$

Let the potentials satisfy the following bounds:

(a) There are constants $D>0$ and $\alpha>2$ such that

$$D(|x|^\alpha-1)\le U(x)\le D(|x|^\alpha+1),\ x\in R^d.$$

(b) There is a decreasing function ϕ on the natural numbers such that

$$\sum_{j\in Z^\nu}\phi(|i-j|)^{1/2}<\infty$$

and

$$|W(x_i,x_j)|\le\phi(|i-j|)\frac{1}{2}(x_i{}^2+x_j{}^2).$$

REMARK. (a) The interaction satisfying the above assumptions is superstable and regular in the sense of Assumption 2. 1. The above assumptions are essentially the same as those in [8].

(b) If the two body potential satisfies the bounds

$$|W(x_i, x_j)| \leq \psi(|i-j|)\frac{1}{2}(|x_i|^\gamma + |x_j|^\gamma),$$

then the constant α in the above assumption (a) can be assumed to be $\alpha > \gamma$. See the proof of Lemma 5. 2. 3.

THEOREM 2. 5. *Let the interaction satisfy* Assumption 2. 3. *There exists* $\beta_0 > 0$ *such that for all* $0 < \beta < \beta_0$ *the followings hold*:

(a) *The infinite volume limit*

$$J_\beta(f) = \lim_{\Lambda \to Z^\nu} J_{\Lambda,\beta}(f), \quad f \in C([0, \beta] \times B, \mathbf{C}^d)$$

exists for all f *as* Λ *tends to* Z^ν. $J(f)$*'s satisfy the cluster property under lattice translations.*

(b) *There exists a unique (up to unitary equivalent) quadruple* ($\hat{\mathscr{H}}_\beta$, $\mathscr{U}$, $\hat{\Omega}_\beta$, $\hat{U}$) *satisfying the properties in* Theorem 2. 4(*b*). *Moreover* $\hat{\Omega}_\beta$ *is invariant under lattice translations and it is only vector in* $\hat{\mathscr{H}}_\beta$ *which has this property.*

We will show the above theorem by using the cluster expansion method developed in [9] and the Wiener integral formalism.

3. Thermodynamic limit of the pressure and equilibrium states

In this section we prove Theorem 2. 2–Theorem 2. 4 using the apriori estimates (proposition 2. 1). In order to show those we will use the Wiener integral formalism together with the methods similar to those in [7, 2, 6]. In Section 3. 1 we introduce the Wiener integral formalism and then we will prove Theorem 2. 2–Theorem 2. 4 in the rest of this section.

3. 1. The Wiener integral formalism

We briefly review the Wiener integral formalism in statistical mechanics. For the details we refer to [2, 6, 15]. The path space of the Wiener measure can be choosen to be

$$\Omega = C([0, \beta], R^d) \qquad (3. 1. 1)$$

where $C([0,\beta], R^d)$ is the family of continuous functions from $[0,\beta]$ to R^d. The Wiener measure $P_\sigma^\beta(x,y;d\omega)$ with diffusion coefficient $\sigma>0$, conditioned on those paths $\omega\in\Omega$ with $\omega(0)=x$, $\omega(\tau=\beta)=y$, is σ-additive, finite measure on Ω. It is the path space measure of the process with transition function given by the kernel of $\exp[\frac{1}{2}\sigma t\Delta]$ denoted by

$$P_\sigma^t(x, y)=\frac{1}{(2\pi\sigma t)^{d/2}}\exp[-\frac{|x-y|^2}{2\sigma t}] \tag{3.1.2}$$

where $|x-y|$ is the Euclidean distance between y and y.

Let H be a self-adjoint operator on $L^2(R^d, dx)$ given by

$$H=-\frac{1}{2}\sigma\Delta+V \tag{3.1.3}$$

where V be a real function on R^d. If Δ and V have a common dense domain, the Feynman-Kac formula gives us

$$e^{-\beta H}(x, y)=\int P^\beta(x, y;d\omega)\exp[-\int_0^\beta V(\omega(\tau))d\tau] \tag{3.1.4}$$

where $e^{-\beta H}(x,y)$ is the kernel of $e^{-\beta H}$. We now apply the above formalism to our models. We write

$$P^\beta(x, y;d\omega)=P^\beta{}_{\sigma=1}(x, y;d\omega)$$

and for $B\subset\Lambda$

$$P_\Lambda{}^\beta(x_\Lambda, y_\Lambda;d\omega_\Lambda)=\prod_{i\in\Lambda}P^\beta(x_i, y_i;d\omega_i) \tag{3.1.5}$$
$$P_{\Lambda,\sigma(B)}{}^\beta(x_\Lambda, y_\Lambda;d\omega_\Lambda)=\prod_{i\in\Lambda\setminus B}P^\beta(x_i, y_i;dx_i)\prod_{j\in B}P_{\sigma_j}{}^\beta(x_j, y_j;d\omega_j).$$

By the Feynman-Kac formula, see e.g. [1, 2], we have

$$Tr(e^{-\beta\mathscr{H}_\Lambda})=\int dx_\Lambda\int P_\Lambda{}^\beta(x_\Lambda, x_\Lambda;d\omega_\Lambda)\exp[-\int_0^\beta U\omega_\Lambda(\tau))d\tau] \tag{3.1.6}$$

and

$$Tr(\exp[-\beta(H_\Lambda+\frac{1}{4}\Delta(B))]\exp(\beta\in x^2(B))) \tag{3.1.7}$$
$$=\int dx_\Lambda\int P_{\Lambda,\sigma(B)}{}^\beta(x_\Lambda,x_\Lambda;d\omega_\Lambda)\exp(\beta\in x^2(B))\exp[-\int_0^\beta U(\omega_\Lambda(\tau))d\tau]$$

where $\sigma_i=1/2$ for $i\in B$. From now on we will use the following abbreviated notations to avoid typographical complications:

$$\int d\mu_\sigma{}^\beta(x,\omega)\equiv\int dx\int P_\sigma{}^\beta(x,x;d\omega)$$
$$\int d\mu_{\Lambda,\sigma(B)}{}^\beta(x_\Lambda,\omega_\Lambda)\equiv\int dx_\Lambda\int P_{\Lambda,\sigma(B)}{}^\beta(x_\Lambda,x_\Lambda;d\omega_\Lambda) \tag{3.1.8}$$

and

$$U(\omega_\Lambda)\equiv\int_0^\beta U(\omega_\Lambda(\tau))\,d\tau$$
$$\omega^2(B)\equiv\sum_{i\in B}\int_0^\beta \omega_i{}^2(\tau)\,d\tau \tag{3.1.9}$$
$$W(\omega_{\Lambda_1},\omega_{\Lambda_2})\equiv\int_0^\beta W(\omega_{\Lambda_1}(\tau),\omega_{\Lambda_2}(\tau))\,d\tau.$$

Furthermore, we will drop β and x in $d\mu^\beta(x,\omega)$ if there is no confusion involved.

3.2. Thermodynamic limit of the pressure

We will use Propsoition 2.1 following the method similar to that used in [7] to show Theorem 2.2. Since the process of the proof is quite similar to that in Section 2 of [7], we will produce necessary bounds and only give a sketch of the proof.

We first note that by Assumption 2.1(e)

$$Z_{\Lambda,\beta}>\lambda^{|\Lambda|} \tag{3.2.1}$$

and so

$$\lambda\leq P_{\Lambda,\beta}\leq\lambda'.$$

Here we have used Assumption 2.1(c) to get the second inequality.

LEMMA 3.2.1. *There is a constant* c *independent of* B *and* Λ *such that for* $B\subset\Lambda$ *and sufficiently small* ε

$$Z_\Lambda{}^{-1}\int d\mu(x_\Lambda,\omega_\Lambda)\exp[-U(\omega_\Lambda)+\varepsilon\omega^2(B)]\leq e^{c|B|}$$

with the notations in (3.1.8) *and* (3.1.9).

Proof. Notice that

$$\int d\mu(x_\Lambda,\omega_\Lambda)\exp[-U(\omega_\Lambda)+\varepsilon\omega^2(B)]$$

$$=Tr(\exp[-\beta(H_\Lambda-\varepsilon x^2(B))])$$
$$\leq Tr(\exp(\beta\varepsilon x^2(B))\exp[-\beta H_\Lambda]).$$

Here we have used the Golden–Thompson inequality [11, 15]. The lemma follows from the above inequality and Proposition 2. 1.

We introduce bad configurations of paths

$$B_\Lambda(N^2, \Delta)=\{\omega_\Lambda\in\Omega^\Lambda : \omega^2(\Delta)\geq N^2|\Delta|\} \tag{3.2.3}$$

where $\omega^2(\Delta)$ has been defined in (3. 1. 9) for $\Delta\subset\Lambda$.

LEMMA 3. 2. 2. *There are constant $\gamma>0$ and δ such that*

$$Z_\Lambda^{-1}\int_{B_\Lambda(N^2,\Delta)}d\mu(x_\Lambda, \omega_\Lambda)e^{-U(\omega_\Lambda)}\leq\exp[-(\gamma N^2-\delta)|\Delta|].$$

Proof. On $B_\Lambda(N^2, \Delta)$ we have $0\leq\varepsilon\omega^2(\Delta)-\varepsilon N^2|\Delta|$. Multiply the factor $\exp[\varepsilon(\omega^2(\Delta)-N^2|\Delta|)]$ to the expression in the lemma. Then the lemma follows from Lemma 3. 2. 1.

Sketch of proof of Theorem 2. 2. The proof will follow from Lemma 3. 2. 1 and Lemma 3. 2. 2, and from the method used in Section 2 of [7] together with the following replacements:

$$\left.\begin{array}{l} s^2 \\ (R^d)^\Lambda \\ d\mu(s) \\ \beta U(s_\Lambda) \end{array}\right\} \text{ by } \left\{\begin{array}{l} \omega^2 \\ \Omega^\Lambda \\ d\mu^\beta(x, \omega) \\ U(\omega_\Lambda). \end{array}\right. \tag{3.2.4}$$

In more explicit we note that for $\Delta_1\cap\Delta_2=\phi$

$$W(\omega_{\Delta_1}, \omega_{\Delta_2})\leq\frac{1}{2}\sum_{i\in\Delta_1}\omega_i^2\Big(\sum_{j\in\Delta_2}\phi(|i-j|)\Big)+\frac{1}{2}\sum_{j\in\Delta_2}\omega_j^2\Big(\sum_{i\in\Delta_1}\phi(|i-j|)\Big)$$

by Assumption 2. 1(c).

As in Section 2 of [7] we introduce the following notations: Let $V_i(\Delta_1, \Delta_2)$, $i=1, 2, \ldots$ be the values taken by the function $\sum_{j\in\Delta_2}\phi(|k-j|)$ when $k\in\Delta_1$, ordered in decreasing order. Let

$$g_i^\Lambda(\Delta_1, \Delta_2)=\{\omega_\Lambda : \sum_{j\in\bar{V}_i(\Delta_1,\Delta_2)}\omega_j^2\leq N^2|\bar{V}_i(\Delta_1, \Delta_2)|\} \tag{3.2.5}$$

where

$$\bar{V}_i(\Delta_1, \Delta_2) = \{k \in \Delta_1 : \sum_{j\in\Delta_2} \phi(|k-j|) \geq V_i(\Delta_1, \Delta_2)\} \tag{3.2.6}$$

and let $g^\Lambda(\Delta_1, \Delta_2) = \bigcap^{\infty} g_i^\Lambda(\Delta_1, \Delta_2)$ (3.2.7)

g_Λ^c = complement of $g^\Lambda(\Delta_1, \Delta_2) \cap g^\Lambda(\Delta_2, \Delta_1)$.

A direct consequence of the above notations is the following:

LEMMA 3.2.3. *Let $\Delta_1 \cap \Delta_2 = \phi$, $\Delta_1, \Delta_2 \subset \Lambda$. We get*

$$|W(\omega_{\Delta_1}, \omega_{\Delta_2})| \leq N^2 \sum_{j\in\Delta_1} \sum_{i\in\Delta_2} \phi(|i-j|)$$

for $\omega_\Lambda \in g^\Lambda(\Delta_1, \Delta_2) \cap g^\Lambda(\Delta_2, \Delta_1)$.

LEMMA 3.2.4. *Let Δ_1 and Δ_2 be as in Lemma 3.2.3. Then there is N independent of $\Lambda, \Delta_1, \Delta_2$ such that*

$$Z_\Lambda^{-1} \int_{g_\Lambda^c} d\mu(x_\Lambda, \omega_\Lambda) e^{-U(\omega_\Lambda)} \leq 1/4.$$

Proof. By Lemma 3.2.2 and (3.2.5) the expression in the lemma is bounded by

$$\begin{aligned} &2\sum_i Z_\Lambda^{-1} \int_{g_i^c(\Delta_1, \Delta_2)} d\mu(x_\Lambda, \omega_\Lambda) e^{-U(\omega_\Lambda)} \\ &\leq 2\sum_j \exp[-(\gamma N^2 - \delta) j]. \end{aligned}$$

For sufficiently large N the above is bounded by $1/4$.

Theorem 2.2 now follows from Lemma 3.2.2–Lemma 3.2.4 and from the method used in Section 2 of [7]. For the details we refer to [7].

3.3. Infinite volume equilibrium states and physical Hilbert spaces

In this section we prove Theorem 2.3 and Theorem 2.4 using the Green's functions method similar to that in Chapter 6.3.4 of [2]. We first begin to prove Theorem 2.3. For $B \subset \Lambda$, let $\{f_{l,j} : j \in B, l = 1, 2, \ldots, d\}$ be an orthonormal base of the finite dimensional Hilbert space $(\mathbf{C}^d)^B$. We define

$$\begin{aligned} \Phi(f_{l,j}) &= x_j^l \\ \Phi(i f_{l,j}) &= -i\partial/\partial x_j^l \end{aligned} \tag{3.3.1}$$

where $x_j{}^l$ and $-i\partial/\partial x_j{}^l$ be the position and momentum operators in the l-coordinate direction at the lattice site $j\in B$. Now if f has the decomposition

$$f=\sum_{j\in B}(s_j{}^l+it_i{}^l)f_{l,j}, \tag{3.3.2}$$

we define the Weyl operator by

$$\begin{aligned} &W(f)=\exp(i\Phi(f)) \\ &\Phi(f)=\sum_{j\in B}\sum_{l=1}^{d}(s_j{}^l x_j{}^l-it_j{}^l\partial/\partial x_j{}^l). \end{aligned} \tag{3.3.3}$$

The algebra $\mathcal{A}_B$ of local observables is generated by $W(f)'s$. Let

$$\mathcal{D}_B=\text{the set of finite linear combinations of } W(f)'s.$$

Then by the CCR (the canonical commutation relations) $\mathcal{D}_B$ is dense in $\mathcal{A}_B$, and $\{\mathcal{D}_B : B\subset Z^\nu,\ B \text{ finite}\}$ forms an isotonic family in the sense of [2].

We prove Theorem 2.3 in the following manner [2]: Let

$$G(A,B;t),\quad A,B\in\mathcal{A},\quad t\in R$$

be the infinite volume Green's functions by a convergent subset Λ_α as in Section 2. We assume that $G:\mathcal{A}\times\mathcal{A}\times R\to\mathbf{C}$ satisfies the following properties:

(1) $A,B\to G(A,B;t)$ is bilinear for all $t\in R$
(2) $t\to G(A,B;t)$ is continuous for all $A,B\in\mathcal{A}$
(3) $G(A,CB;0)=G(AC,B;0)$ for all $A,B,C\in\mathcal{A}$
(4) $G(\mathbb{1},\mathbb{1};0)=1$
(5) $\sum_{i,j=1}^{n}G(A_i{}^*,A_j;t_j-t_i)\geq 0$ for any finite sequence $\{A_i\}_{i=1}{}^n$ in $\mathcal{A}$ and $\{t_i\}_{i=1}{}^n$ in R
(6) [Weak KMS condition] For all $A,B\in\mathcal{A}$, and for all $\hat{f}\in\mathcal{D}$

$$\int dt\, f(t)G(A,B;t)=\int dt\, f(t+i\beta)G(A,B;-t).$$

Then Theorem 2.3 follows from Theorem 6.3.27 and Theorem 6.3.28 of [2]. Since the properties (1) and (3)-(5) easily follow from those of $G_\Lambda(A,B;t)$, the following lemma together with proposition 6.3.29 of [2] shows that $G(A,B;t)$ also satsfies (2) and (6).

LEMMA 3.3.1. *Let* $A\in\mathcal{D}_B$, $B\subset\Lambda$. *There exists a constant* c_A *inde-*

pendent of Λ such that

$$\omega_\Lambda([H_\Lambda, A]^*[H_\Lambda, A]) \leq c_A < \infty.$$

Proof. Since $A \in \mathcal{D}_B$ is a finite linear combination, of $W(f)'s$, it suffices to prove the lemma for $A = W(f)$, $f \in (\mathbf{C}^d)^B$. In the following every identity is first defined as bilinear forms on a dense domain and then as operators on an appropriate dense domain if one side makes sense. We will not give justifications of domain problems explicitly and leave those to the reader. Notice that

$$[H_\Lambda, W(f)] = [-\frac{1}{2}\Delta(\Lambda), W(f)] + [U(x_\Lambda), W(f)]. \tag{3.3.4}$$

Since $W(f) \in \mathcal{A}_B$ it follows that

$$\begin{aligned} &[-\frac{1}{2}\Delta(\Lambda), W(f)] = [-\frac{1}{2}\Delta(B), W(f)] \\ &[U(x_\Lambda), W(f)] = [U(x_B) + W(x_B, x_{\Lambda \setminus B}), W(f)]. \end{aligned} \tag{3.3.5}$$

We will use the following formula: Let A, B be (unbounded) operators and let B be self-adjoint. Define

$$\begin{aligned} &\Gamma^0[B](A) = A, \Gamma^1[B](A) = [B, A] \\ &\Gamma^{n+1}[B](A) = [B, \Gamma^n[B](A)]. \end{aligned} \tag{3.3.6}$$

It then follows that

$$\begin{aligned} [A, e^{iB}] = &\sum_{m=1}^{n} \frac{1}{m!} e^{iB} \Gamma^m[-iB](A) \\ &+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_n} ds_{n+1} e^{i(1-s_{n+1})B} \Gamma^{n+1}[-iB](A) e^{is_{n+1}B} \end{aligned}$$

provided the expressions make sense.
The above identity follows from

$$Ae^{isB} - e^{isB}A = \int_0^s ds_1 e^{i(s-s_1)B}[-iB, A] e^{is_1 B}$$

and from an induction.

We now turn to estimates for (3.3.5). We observe that

$$\omega_\Lambda(\sum_{j \in B} x_j^{2m}) \leq c_1(m, B) \tag{3.3.8}$$

and

$$\omega_\Lambda(-\Delta) \leq c_2(B) \tag{3.3.9}$$

for some constant c_1 and c_2 independent of Λ. The bound (3.3.8) follows directly from Proposition 2.1. To show (3.3.9) one notices that $\mathrm{Tr}(\exp[-\beta(H_\Lambda+z\Delta(B))])$ is analytic for $|z|<1/2$. Then the bound (3.3.9) follows from the Cauchy integral formula, Proposition 2.1 and the property of ω_Λ (defined by trace). By the abstract Hölder inequality [11] (or the Schwarz inequality for the state ω_Λ) it is easy to check that

$$|\omega_\Lambda(AB)|\leq\omega_\Lambda(A^*A)^{1/2}\omega_\Lambda(B^*B)^{1/2} \tag{3.3.10}$$

for unbounded operators A, B whenever the *r. h. s.* is finite.

Let $W(f)$ and $\Phi(f)$ bedefined as in (3.3.3). From the CCR it follows that

$$[\Phi(f), -\Delta(B)]=2i\sum_{j\in B}\sum_{l=1}^{d} s_j{}^l\partial/\partial x_j{}^l \qquad (3.3.11)$$

$$[\Phi(f), [\Phi(f), -\Delta(B)]]=-2\sum_{j\in B}\sum_{l=1}^{d}(s_j{}^l)^2 \qquad (3.3.11)$$

and $\Gamma^n[\Phi(f)](\Delta(B))=0$ if $n\geq3$, and

$$\Gamma^m[\Phi(f)](\sum_{j\in B}|x_j|^{2n})=(-i)^m\sum_{j\in B}\sum_{l=1}^{d}(t_j{}^l)^m(x_j{}^l)^{2n-m}2n!/(2n-m)! \tag{3.3.12}$$

For $m>2m$, the r. h. s. of (3.3.12) becomes zero. We now use (3.3.7), (3.3.11), (3.3.10) and (3.3.9) (in that order) to obtain

$$\begin{aligned}&\omega_\Lambda([-\tfrac{1}{2}\Delta(B), W(f)]^*[-\tfrac{1}{2}\Delta(B), W(f)])\\&\quad\leq c_1'(f,B)\omega_\Lambda(-\Delta(B)) +c_2'(fmB)\\&\quad\leq c(f,B).\end{aligned} \tag{3.3.13}$$

Next we use (3.3.10), Assumption 2.2. (3.3.7), (3.3.12) and (3.3.8) (in that order) to get

$$\begin{aligned}&\omega_\Lambda([U(x_B), W(f)]^*[U(x_B), W(f)]\\&\quad\leq2\omega_\Lambda(U(x_B)^2)+2\omega_\Lambda(W(f)^*U(x_B)^2W(f))\\&\quad\leq4\omega_\Lambda((D\sum_{j\in B}x_j{}^{2n})^2)+4\omega_\Lambda(W(f)^*(D\sum_{j\in B}x_j{}^{2n})^2W(f)\\&\qquad+c_1''(f,B)\\&\quad\leq c_2''(f,B)(\sum_{j\in B}\omega_\Lambda(x_j{}^{4n}))+c_3''(f,B)\\&\quad\leq c''(f,B).\end{aligned} \tag{3.3.14}$$

We note that for any $i \in B$

$$\sum_{j\in \bar{B}\setminus B} \phi(|i-j|)\omega_\Lambda(x_j^2) \le \text{const} \tag{3.3.15}$$

by Proposition 2.1 (or by (3.3.8)). We now use Assumption 2.1(d), (3.3.15) and the method similar to that used to get (3.3.14) to obtain

$$\omega_\Lambda([W(x_B, x_{\Lambda\setminus B}), W(f)]^*[W(x_B, x_{\Lambda\setminus B}), W(f)]) \le c'''(f, B). \tag{3.3.16}$$

Now Lemma 3.3.1 follows from (3.3.4), (3.3.5), (3.3.13), (3.3.14) and (3.3.16).

We have completed the proof of Theorem 2.3. We next turn to show Theorem 2.4. Let $\phi(f)$ and $J_{\Lambda,\beta}(f)$ be defined as in (2.17) and (2.18), respectively. We first prove the bounds given in (2.19).

LEMMA 3.3.2 *Let $f \in C([0,\beta]\times B; \mathbf{C}^d)$ for finite $B \subset \Lambda$. There exists constant c independent of Λ and f such that*

$$|J_{\Lambda,\beta}(f)| \le \exp(c|||f|||^2)$$

where $|||f||| = \sum_{j\in B} \left(\int_0^\beta |f(\tau)i)|^2 d\tau\right)^{1/2}$.

Proof. Let $dp_{\Lambda,\beta}$ be the probability measures defined as in (2.16). For $i \in B$, let $\|\omega_i\|_2^2 = \int_0^B \omega_i(\tau)^2 d\tau$. From Lemma 3.2.1 it follows that

$$\int \|\omega_i\|_2^{2n} dP_{\Lambda,\beta} \le a\, n! \tag{3.3.17}$$

for some constant a. From the definition of $\phi(f)$ in (2.17) we have

$$\begin{aligned} |\phi(f)| &\le \sum_{i\in B} \int_0^\beta |\omega_i(\tau) f(\tau, i)| d\tau \\ &\le \sum_{i\in B} \|\omega_i\|_2 \, \|f(i)\|_2 \end{aligned} \tag{3.3.18}$$

where $\|f(i)\|^2 = \int_0^\beta |f(\tau, i)|^2 d\tau$. We write

$$\phi(f(i)) = \int_0^\beta \omega_i(\tau) f(\tau, i) d\tau.$$

We then have

$$|\int\phi(f)^n dP_{\Lambda,\beta}| \leq \sum_{i_1\in B}\cdots\sum_{i_n\in B}|\int\prod_{k=1}^{n}\phi(f(i_k))dP_{\Lambda,\beta}|.$$

We use Hölder inequality, (3. 3. 18) and (3. 3. 17) to obtain

$$|\int\prod_{k=1}^{n}\phi(i_k)\,dP_\Lambda| \leq \prod_{k=1}^{n}\|f(i_k)\|_2 \left(\int\|\omega_{i_k}\|_2^{2n}dP_\Lambda\right)^{1/2n}$$
$$\leq b^n\sqrt{n!}\prod_{k=1}^{n}\|f(i_k)\|_2.$$

Thus it follows that

$$|\int\phi(f)^n dP_{\Lambda,\beta}| \leq b^n\sqrt{n!}\left(\sum_{i\in B}\|f(i)\|_2\right)^n.$$

The lemma follows from the above bounds.

Proof of Theorem 2. 4. The generating functional $J_\beta(f)$ obviously satisfies the properties (a) and (c)-(e) in the above Theorem 2. 4 in Section 2. The continuity (b) of $J_\beta(f)$ in the norm $|||\cdot|||$ follows from a consequence of Lemma 3. 3. 2. The first part of the theorem follows from Minlo's theorem [15]. To show the part (b) we can use the reconstruction process used in constructive quantum field theory (see e. g. [6, 16] to construct $\mathscr{H}_\beta$, $\mathscr{U}$, $\hat{\Omega}_\beta$ and $\hat{U}(t)$ in the theorem. The KMS condition follows from the definition of $dP_{\Lambda,\beta}$ and from the property of finite volume Gibbs states. The modularity of ω can be shown by the method used in Example 5. 3. 13 of [2]. This completes the proof of Theorem 2. 4.

4. Proof of apriori estimates

This section is devoted to the proof of Proposition 2. 1. We first derive a result which we will use later.

PROPOSITION 4. 1. *Let $\alpha>0$ and $A>0$. For small $\varepsilon>0$ (depending on β, σ and A) there exists a constant c such that*

$$Tr_{L^2(R^d)}(\exp(\beta\varepsilon x^2)\exp(-\beta[-\tfrac{1}{2}\sigma\Delta+Ax^2])\leq c.$$

REMARK. Obviously the above bound is equivalent to

$$\int dx e^{\beta\varepsilon x^2}\int_\sigma^\beta P(x,x;d\omega)e^{-A\omega^2}\leq d \tag{4. 1}$$

where $\omega^2=\int_0^\beta \omega(\tau)^2 d\tau$.
The method we will use to show the proposition is quite similar to that used in the proof of Proposition III. 2. 2 of [10]. The reader to whom proposition 4. 1 seems to be trivial may skip the proof in the following.

Before proving the proposition we need technical lemmas. Let $A\subset R^d$ be a regular compact and let

$$w_A(x, y; \beta)=\int P_\sigma^\beta(x, y; d\omega)[1-\chi_A(\omega)] \tag{4.2}$$

where χ_A be the characteristic function of the set $\{\omega : \omega(\tau)\in A, 0\leq\tau\leq \beta\}$.

LEMMA 4. 2. *For all* $x, y\in A$, $\sigma>0$

$$0\leq w_A(x, y; \beta)\leq\frac{1}{(2\pi\sigma\beta)^{\nu/2}}\exp\left[-\frac{1}{2\sigma\beta}d(x, \partial A)^2\right]$$

where $d(x, A)$ *is the distance from* x *to* ∂A *(the boundary of* A*).*

Proof. See Theorem 6. 3. 8 of [2] and its proof.
For any $x\in R^d$ we write

$$V(x, \omega)\equiv\max_{0\leq\tau\leq\beta} \operatorname{dist}(x, \omega(\tau)). \tag{4.3}$$

LEMMA 4. 3. *Let* B *be a ball in* R^d. *Then there is a constant* c *independent of* B *such that*

$$\int_B dx\int P_\sigma^\beta(x, x; d\omega)\exp[\frac{1}{4\sigma\beta}V(x, \omega)^2\leq c|B|$$

where $|B|$ *is the volume of* B.

Proof. For a given $x\in R^d$ we decompose the space of paths to disjoint subsets as the following:

$$\begin{aligned} &\Omega=\bigcup_{l=0}^{\infty}\varepsilon_l \\ &\varepsilon_l=\{\omega : l\leq V(x, \omega)<l+1\},\quad l=0, 1, 2, \ldots \end{aligned} \tag{4.4}$$

Let

$$\varepsilon_{(\geq l)}=\{\omega; V(x, \omega)\geq l\}. \tag{4.5}$$

Then the expression in the lemma equals to sum of $E_l(B)$, where

$$E_l(B)\equiv\int_B dx\int_{\varepsilon_l} P_\sigma^\beta(x,x;d\omega)\exp\left[\frac{1}{4\sigma\beta}V(x,\omega)^2\right]. \tag{4.6}$$

Since $V(x,\omega)^2\leq(l+1)^2$ on ε_l and since $\varepsilon_l\subset\varepsilon_{(\geq l)}$. it follows that

$$\begin{aligned}E_l(B)&\leq\exp\left[\frac{1}{4\sigma\beta}(l+1)^2\right]\int_B dx\int_{\varepsilon(\geq l)}P_\sigma^\beta(x,x;d\omega)\\&\leq\exp\left[\frac{1}{4\sigma\beta}(l+1)^2\right]\int_B dx\int P_\sigma^\beta(x,x;d\omega)[1-\chi_{A_l}(\omega)]\end{aligned}$$

where $A_l=\{y\in R^d;\ \text{dist}\ (x,y)\leq l\}$. From (4.2) and Lemma 4.2 we conclude that

$$E_l(B)\leq|B|\ \exp\left[\frac{1}{4\sigma\beta}(l+1)^2-\frac{1}{2\sigma\beta}l^2\right] \tag{4.7}$$

and so

$$\sum_{l=0}^{\infty}E_l(B)\leq c|B|$$

for some constant c. This completes the proof of the lemma.

Proof of Proposition 4.1. Let B_l be the ball of radius l centered at the originin R^d. Then

$$\int dx\ldots=\int_{B_1}dx\ldots+\sum_{l=1}^{\infty}\int_{B_{l+1}\setminus B_l}dx\ldots. \tag{4.8}$$

We note that

$$\begin{aligned}x^2&=\frac{1}{\beta}\int_0^\beta[x-\omega(\tau)+\omega(\tau)]^2d\tau\\&\leq\frac{2}{\beta}\int_0^\beta[x-\omega(\tau)]^2d\tau+\frac{2}{\beta}\int_0^\beta\omega(\tau)^2d\tau\\&\leq 2V(x,\omega)^2+\frac{2}{\beta}\omega^2\end{aligned}$$

and so

$$-\omega^2\leq\beta V(x,\omega)^2-\frac{1}{2}\beta x^2. \tag{4.9}$$

Let $m=\min\left\{\frac{1}{4\sigma\beta^2},A\right\}$. We use Lemma 4.3 and (4.9) to obtain

$$\int_{B_{l+1}\setminus B_l}dx\int P_\sigma^\beta(x,x;d\omega)\exp[\beta\varepsilon x^2-A\omega^2]$$

$$\leq \int_{B_{l+1}\setminus B_l} dx \exp[\beta(\varepsilon - \frac{1}{2}m)x^2] \int P_\sigma{}^\beta(x, x; d\omega) \exp[\beta m V(x, \omega)^2]$$
$$\leq C|B|_{l+1} \exp[\beta c\varepsilon - \frac{1}{2}m) l^2] \tag{4.10}$$

if $\varepsilon < \frac{1}{2}m$, and

$$\int_{B_1} dx \exp(\beta\varepsilon x^2) \int P_\sigma{}^\beta(x, x; d\omega) \exp[-A\omega^2]$$
$$\leq \exp(\beta\varepsilon) \int_{B_1} dx \int P_\sigma{}^\beta(x, x; d\omega) \tag{4.11}$$
$$\leq c'.$$

If $\varepsilon < \frac{1}{2}m$, the sum of the *r. h. s.* of (4.10) with respect to l converges. Thus the proposition follows from (4.1), (4.10) and (4.11).

Following the main idea of [14] (with some necessary modifications) we now begin to show Proposition 2.1. Given $\alpha > 0$, we can choose an integer $p_0 > 0$ and for $j \geq p_0$ an integer l_j such that

$$\left| \frac{|l_{j+1}}{l_j} - (1+2\alpha) \right| < \alpha.$$

As in [14] we introduce the notation

$$[j] = \{i \in Z^\nu : |i| \leq l_j\}$$
$$V_j = (2l_j + 1)^\nu.$$

PROPOSITION 4.4. [*Ruelle* [14, 13]] *Let* $\varepsilon' > 0$ *and* $c \geq 0$ *be given, and let* ϕ *be a decreasing function on integers such that*

$$\sum_{i \in Z^\nu} \phi(|i|) < \infty.$$

If α *is sufficiently small, one can choose an increasing sequence* $\langle \phi_j \rangle$ *such that* $\phi_1 \geq 1$, $\phi_i \to \infty$ *and fixed* $p \geq p_0$ *so that the following is true: Let* $n(\cdot)$ *be a function from* Z^ν *to real* ≥ 0. *Suppose there is* q *such that* $q \geq p$ *and* q *is the largest number for which*

$$\sum_{i \in [q]} n(i)^2 \geq \phi_q V_q.$$

Then

$$\sum_{i\in[q+1]} c+\sum_{i\in[q+1]}\sum_{j\notin[q+1]}\phi(|i-j|)\frac{1}{2}(n(i)^2+n(j)^2)$$
$$\leq\varepsilon'\sum_{i\in[q+1]} n(i)^2.$$

We fix an integer p such that the above proposition holds. We decompose the space of path configurations into disjoint subsets:

$$\Omega^{\Lambda}=Q_0\cup(\bigcup_{q\geq p} Q_q)$$

where

$$Q_0=\{\omega\in\Omega^{\Lambda}:\sum_{i\in q[]}\omega_i^2<\phi_q V_q \text{ for all } q\geq p\} \tag{4.12}$$
$$Q_q=\{\omega\in\Omega^{\Lambda}: q \text{ is the largest integer such that } \sum_{i\in[b]}\omega_i^2\geq\phi_q V_q\}.$$

With the notations in (3.1.8) and (3.1.9), let

$$Z_{\Lambda}^{-1}\, Tr(\exp[\beta\varepsilon x^2(B)]\exp[-\beta(H_{\Lambda}+\frac{1}{4}\Delta(B))])$$
$$=Z_{\Lambda}^{-1}\int d\mu_{\Lambda,\sigma(B)}(x_{\Lambda},\omega_{\Lambda})\exp[-U(\omega)+\beta\varepsilon x^2(B)] \tag{4.13}$$
$$=\rho'+\rho''$$

where ρ' corresponds to the contribution from the subset Q_0 and ρ'' corresponds to the contributions from the other subsets Q_q.

PROPOSITION 4.5. *Assume $B\cap[p]\neq\phi$ and $i\in B\cap[p]$. There are positive constants D_1, D_2 and D_3 such that*

$$\rho'\leq D_1 Z_{\Lambda}^{-1}Tr(\exp[\beta\varepsilon x^2(B-\{i\})]\exp[-\beta(H_{\Lambda}+\frac{1}{4}\Delta(B-\{i\}))]) \tag{4.14}$$

and

$$\rho''\leq\sum_{q\geq q}\exp(D_2 V_{q+1}-D_3\phi_{q+1}V_{q+1}) \tag{4.15}$$
$$\cdot Z_{\Lambda}^{-1}Tr(\exp(\beta\varepsilon x^2(B\backslash[q+1]))\exp[-\beta(H_{\Lambda}-\frac{1}{4}\Delta(B\backslash[q+1]))]).$$

Proof of Proposition 2.1. By a translation we may assume that B contains the origin of Z^{ν} and so $B\cap[P]\neq\phi$. Now, Proposition 2.1 follows from an induction on card (B) and from the above proposition.

In the remainder of this section we prove Proposition 4.5.

Proof of Proposition 4. 5. From (3. 1. 7) and from the definition of ρ' in (4. 13) we have

$$\rho'=Z_\Lambda^{-1}\int_{Q_0}d\mu(x_{\Lambda-\{i\}},\omega_{\Lambda-\{i\}})\ d\mu_\sigma(x_i,\omega_i) \cdot \exp(\beta\varepsilon x_i^2-U(\omega_i))\exp(-U(\omega_{\Lambda-\{i\}})-W(\omega_i.\ \omega_{\Lambda-\{i\}})+\beta\varepsilon x^2(B-\{i\})). \tag{4.16}$$

Let Σ be a ***bounded*** Borel set in Assumption 2. 1 (e) and let

$$\mathcal{E}=\{\omega\in\Omega:\omega(\tau)\in\Sigma,\tau\in[0,\beta]\}.$$

Let $\omega_i'\in\mathcal{E}$. For $i\in B\cap[p]$ we have that on Q_0

$$\begin{aligned}-W(\omega_i,\omega_{\Lambda-\{i\}})&=-W(\omega_i',\omega_{V-\{i\}})\\&\quad-\{W(\omega_i,\omega_{\Lambda-\{i\}})-W(\omega_i',\omega_{\Lambda-\{i\}}))\\&\leq-W(\omega_i',\omega_{\Lambda-\{i\}})+\frac{1}{2}(\sum_j\phi(|j|))(\omega_i^2+\omega_i'^2)+D\\&\leq-W(\omega_i',\omega_{\Lambda-\{i\}})+2D\end{aligned}\tag{4.17}$$

for some constant D. Here we have used the definition of Q_0 in (4. 12), Assumption 2. 1 (d) and the fact that $\omega_i'\in\mathcal{E}$. We multiply the factor

$$\lambda\int_{\mathcal{E}}d\mu(x_i',\omega_i')\exp(-U(x_i')\quad(\geq 1\text{ by Assumption 2. 1(e)})$$

the *r. h. s.* of (4. 16) and then we use (4. 17) to conclude that

$$\rho'\leq[\int d\mu_\sigma(x_i,\omega_i)\exp(\beta\varepsilon x_i^2-U(\omega_i))]e^{2D} \cdot Z_\Lambda^{-1}\int du_{\Lambda,\sigma(B-\{i\})}(x_\Lambda,\omega_\Lambda)\ \exp[-U(\omega_\Lambda)+\beta\varepsilon x^2(B-\{i\})].$$

The inequality (4. 14) follows from the above inequality, Assumption 2. 1 (c) and (4. 1) (and (3. 1. 7)).

To prove (4. 15) we note that

$$\rho''=\sum_{q\geq p}Z_\Lambda^{-1}\int_{Q_q}d\mu_{\sigma(B)}(x_\Lambda,\omega_\Lambda)\exp[-U(\omega_{[q+1]\cap\Lambda})+\beta\varepsilon x^2(B\cap[q+1])] \cdot\exp[-U(\omega_{\Lambda\setminus[q+1]})-W(\omega_{[q+1]\cap\Lambda},\omega_{\Lambda\setminus[q+1]})+\beta\varepsilon x^2(B\setminus[q+1])]. \tag{4.18}$$

Let $\omega'_{[q+1]\cap\Lambda}\in\mathcal{E}^{[q+1]\cap\Lambda}$. We note that on Q_q

$$\begin{aligned}&-W(\omega_{[q+1]\cap\Lambda},\omega_{\Lambda\setminus[q+1]})\\&=-W(\omega'_{[q+1]\cap\Lambda},\omega_{\Lambda\setminus[q+1]})+A(\omega,\omega';[q+1],\Lambda)\end{aligned}\tag{4.19}$$

where

$$A(\omega,\omega';[q+1],\Lambda)=-W(\omega_{[q+1]\cap\Lambda},\omega_{\Lambda\setminus[q+1]})+W(\omega'_{[q+1]\cap\Lambda},\omega_{\Lambda\setminus[q+1]})].$$

From Assumption 2.1 (c), Proposition 4.4 and the fact that $\omega'_{[q+1]\cap\Lambda}\in\mathcal{E}^{[q+1]\cap\Lambda}$ it follows that

$$\begin{aligned}
&|A(\omega,\omega';[q+1],\Lambda)|-U(w_{[q+1]\cap\Lambda})|\\
&\le\sum_{i\in[q+1]\cap\Lambda}\sum_{j\in\Lambda\setminus[q+1]}\phi(|i-j|)\left[\frac{1}{2}(\omega_i^2+\omega_j^2)+\frac{1}{2}(\omega_i'^2+\omega_j^2)\right]\\
&\quad-\sum_{i\in[q+1]\cap\Lambda}(A\omega_i^2-C)\\
&\le-\sum_{i\in[q+1]\cap\Lambda}(A-2\varepsilon')\omega_i^2\\
&\le-\sum_{i\in[q+1]\cap\Lambda}\frac{1}{2}(A-2\varepsilon')\omega_i^2-c'\phi_{q+1}V_{q+1}.
\end{aligned}\tag{4.21}$$

To each term in (4.18) corresponding to q we multiply the factor

$$\lambda^{|[q+1]\cap\Lambda|}\int_{\mathcal{E}^{[q+1]\cap\Lambda}}d\mu(x'_{[q+1]\cap\Lambda},\omega'_{[q+1]\cap\Lambda})\exp[-U(\omega'_{[q+1]\cap\Lambda})]$$

(≥ 1 by Assumption 2.1 (d) and (3.1.5))

and then we use (4.19) and (4.20) to conclude that

$$\begin{aligned}
\rho''\le&\sum_{q\ge p}\lambda^{|[q+1]\cap\Lambda|}e^{-c'\phi_{q+1}V_{q+1}}\\
&\cdot\left[\int d\mu_{\sigma(B\setminus[q+1])}(x_\Lambda,\omega_\Lambda)\exp[-U(\omega_\Lambda)+\beta\varepsilon x^2(B\setminus[q+1])]\right]\\
&\cdot\left\{\int d\mu_{\sigma(B\cap[q+1])}(x_{[q+1]\cap\Lambda},\omega_{[q+1]\cap\Lambda})\right.\\
&\left.\cdot\exp[-\frac{1}{2}(A-2\varepsilon')\omega^2_{[q+1]\cap\Lambda}+\beta\varepsilon x^2(B\cap[q+1])]\right\}.
\end{aligned}$$

The last bracket $\{\ldots\}$ in the above expression is bounded by $\exp(c|[q+1]\cap\Lambda|)$ by (4.1) and by the factorization property of the bracket. Thus (4.15) follows from the above inequality and (3.1.7) This completes the proof of Proposition 4.5.

5. The cluster expansion: Proof of Theorem 2.5

In this section we prove Theorem 2.5 (a) using the cluster expansion method developed in [9] together with the Wiener integral formalism. Since Theorem 2.5(b) follows as a consequence of the part (a) of the theorem and Theorem 2.4, it suffices to show Theorem 2.5(a). The cluster expansion for our models is reviewed in Section 5.1. In Section

5.2 we prove Theorem 2.5(a) by showing the convergence of the cluster expansion. We will prove necessary bounds which guarantee the convergence according to [9] and give a sketch of the proof. Throughout this section we denote f as a bounded $\mathbf{C}^d$-valued function on $[0, \beta]\times B$; $f\in C([0,\beta]\times B, \mathbf{C}^d)$, and $\phi(f)$ and $J_{\Lambda,\beta}(f)$ defined as in (2.17) and (2.18), respectively.

5.1. The cluster expansion

From the definitions in (2.16) and (2.18), the generating functional can be written as

$$J_{\Lambda,\beta}(f)=Z_{\Lambda,\beta}{}^{-1}\int d\mu_\Lambda(x_\Lambda,\omega_\Lambda)e^{i\phi(f)}e^{-U(\omega_\Lambda)} \qquad (5.1.1)$$

$$\phi(f)=\sum_{i\in B}\int_0^\beta \omega_i(\tau)f(\tau,i)\,d\tau.$$

We will develop the cluster expansion for $J_{\Lambda,\beta}(f)$.

We review briefly the cluster expansion for our models, which is essentially the same as that in section 2 of [9]. For each $\{i,j\}=X\subset\Lambda$ we assign a real number $s(X), 0\leq s(X)\leq 1$. We write

$$T(\Lambda)=\{X\subset\Lambda : \mathrm{card}(X)=2\} \qquad (5.1.2)$$

$$\{s\}_{T(\Lambda)}=\{s(X) : X\in T(\Lambda)\}.$$

For given $\{s\}_{T(\Lambda)}$ we define

$$W(\omega_\Lambda, \{s\}_{T(\Lambda)})=\sum_{\{i,j\}=X\Lambda} s(X)\,W(\omega_i,\omega_j)$$

$$U(\omega_\Lambda, \{s\}_{T(\Lambda)})=\sum_{i\in\Lambda}U(\omega_i)+W(\omega_\Lambda, \{s\}_{T(\Lambda)}) \qquad (5.1.3)$$

where we have used the notation in (3.1.9). We also write

$$Z_{\Lambda,\beta}(\{s\}_{T(\Lambda)})=d\mu_\Lambda(x_\Lambda,\omega_\Lambda)\exp(-U(\omega_\Lambda, \{s\}_{T(\Lambda)}))$$

$$J_{\Lambda,\beta}(f, \{s\}_{T(\Lambda)})=[Z_{\Lambda,\beta}(\{s\}_{T(\Lambda)})]^{-1}\int d\mu_\Lambda e^{i\phi(f)}\exp[-U(\omega_\Lambda, \{s\}_{T(\Lambda)})]. \qquad (5.1.4)$$

We define

$$Z_\beta{}^{(0)}=Tr_{L^2(R^d)}(\exp[-\beta(-\tfrac{1}{2}\Delta+U(x))]) \qquad (5.1.5)$$

$$\hat{Z}_{\Lambda,\beta}=(Z_\beta{}^{(0)})^{-|\Lambda|}Z_{\Lambda,\beta}.$$

From the above notations it follows that for $\{s\}_{T(\Lambda)}=\{1\}_{T(\Lambda)}$ we recover original partition function and the generating functionals in (5.1.1), and for $\{s\}_{T(\Lambda)}=\{0\}_{T(\Lambda)}$ we have completely decoupled theory.

We may first study the following expression:

$$f_\Lambda(\{s\}_{T(\Lambda)})=\int d\mu_\Lambda \exp(-U_\Lambda, \{s\}_{T(\Lambda)})e^{i\phi(f)}. \tag{5.1.6}$$

For a given $\{i, j\}=X\in T(\Lambda)$ we define

$$\begin{aligned}&\delta^X f(\{s\}_{T(\Lambda)})=f(\{s\}_{T(\Lambda)-X}, \{1\}_X)-f(\{s\}_{T(\Lambda)-X}, \{0\}_X)\\&\varepsilon^X f(\{s\}_{T(\Lambda)})=f(\{s\}_{T(\Lambda)-X}, \{0\}_X).\end{aligned} \tag{5.1.7}$$

For given $B, Y\subset\Lambda$ we define

$$K^\beta(B, Y, f_{B\cup Y})=[Z_\beta^{(0)}]^{-|B\cup Y|} \sum_{\substack{\{X_1,\cdots,X_m\}\subset T(\Lambda)\\ \cup X_i=Y\\ [B, X_1,\dots,X_m\}\text{ connected.}}} [\prod_{i=1}^m \delta^{X_i} f_{B\cup Y}(\{0\}_{T(B\cup Y)})] \tag{5.1.8}$$

Following the steps in section 2 of [9] we obtain:

THEOREM 5.2.1. *The following identity holds:*

$$J_{\Lambda,\beta}(f, \{s\}_{T(\Lambda)})=\sum_{\substack{Y\subset\Lambda\\ B\cap Y\neq\phi}} K^\beta(B, Y, f_{B\cup Y})\frac{\hat{Z}_{\Lambda\setminus B\cup Y,\beta}}{Z_{\Lambda,\beta}}$$

where $f\in C([0,\beta]\times\beta, \mathbf{C}^d)$.

The above identity is called the cluster expansion. We do not produce the proof of the above theorem and refer to section 2 of [9].

We next consider the following expression:

$$g^{\Lambda\beta}(Y)\equiv\frac{\hat{Z}_{\Lambda\setminus Y,\beta}(\{1\}_{T(\Lambda\setminus Y)})}{Z_{\Lambda,\beta}(\{1\}_{T(\Lambda)})}. \tag{5.1.9}$$

Let f be a function on the family of finite subsets of Z^ν. Such functions form a Banach space of the form

$$\mathcal{F}_\xi=\{f: \|f\|=\sup_Y \xi^{-|Y|}|f(Y)|<\infty\} \tag{5.1.10}$$

for a given $\xi>0$. For a given $i_0\in Z^\nu$ we define an operator K on $\mathcal{F}_\xi$ by

$$\begin{aligned}&(Kf)(\phi)=0\\&(Kf)(Y)=f(Y-\{i_0\})-\sum_{\substack{\phi\neq S\subset Z^\nu\setminus Y-\{i_0\}\\ i_0\in S}} K^\beta(\{i_0\}, S;\hat{Z}_{\{i_0\}\cup S})f(Y\cup S)\end{aligned} \tag{5.1.11}$$

where $K^{\beta}(\{i_0\}, Y; \hat{Z}_{\{i_0\}\cup Y})$ is defined as in (5.1.8). For $\phi \neq \Lambda \subset Z^{\nu}$ we introduce the operator χ_{Λ} on $\mathfrak{F}_{\xi}$ by

$$(\chi_{\Lambda} f)(Y) = \chi_{\Lambda}(Y) f(Y) \quad \text{for all } f \in \mathfrak{F}_{\xi}$$

where $\chi_{\Lambda}(Y)=1$ if $Y \subset \Lambda$ and $\chi_{\Lambda}(Y)=0$ otherwise. Let $\mathbb{1}$ be the element in $\mathfrak{F}_{\xi}$ defined by $\mathbb{1}(\phi)=1$ and $\mathbb{1}(Y)=0$ otherwise. Following the steps from (4.5) to (4.8) in [9] and using Theorem 5.1.1 one may easily prove the following result:

LEMMA 5.2.2. $g_{\Lambda}{}^{\beta} = \mathbb{1} + \chi_{\Lambda} K \chi_{\Lambda} g_{\Lambda}{}^{\beta}$.

The above relation is the integral equation of the Kirwood-Salsburg type. See section 4 of [9] for the detailed derivation.

5.2. Convergence of the cluster expansion

In this section we prove the convergence of the cluster expansion and the cluster property of generating functionals. As we said before, we do not produce the detailed proof. We will derive necessary bounds which ensure the convergence of the expansion, and give a sketch of the proof. We first state the main result in this section.

THEOREM 5.2.1. *Assume the interaction satisfies* Assumption 2.3. *For sufficently small $\beta>0$, the cluster expansionin* Theorem 5.1.1 *converges absolutely and uniformly in Λ. Furthermore, there exist constants c and $B(\beta)$ such that for $f \in C([0,\beta] \times B, \mathbf{C}^d)$*

$$|J_{\Lambda,\beta}(f, \{1\}_{T(\Lambda)})| \leq \exp(c|||f|||) e^{c|B|} B(\beta)$$

where $B(\beta) \to 1$ as $\beta \to 0$.

In order to show the above theorem we need the following results:

LEMMA 5.2.2. *Under the assumptions as in* Theorem 5.3.1, *the followings hold*:

(a) $|g_{\Lambda}{}^{\beta}(Y)| \leq \exp(c\beta|Y|)$

for some constant c independent of Λ and β.

(b) *The limit* $g^{\beta} = \lim\limits_{\Lambda \to Z^{\nu}} g_{\Lambda}{}^{\beta}$ *exists.*

LEMMA 5.2.3. *Let $A'>0$. Then there exist constants A'' and c inde-*

pendent of β such that

$$(Z_\beta^{(0)})^{-1}\int d\mu^\beta(x,\omega)\omega^2 \exp(-A'U(\omega))\leq A''|\beta|^{(\alpha-2)/\alpha}e^{c\beta}.$$

We postpone the proofs of Lemma 5.3.2 and 5.3.3 to the end of this section.

Sketch of Proof of Theorem 5.2.1. We assume that Lemma 5.2.2 and Lemma 5.2.3 hold. The basic tools we use are the fundamental theorem of calculus and an induction argument used in section 3 of [9]. We note that

$$\delta^X f(\{s\}_{T(\Lambda)-X})=\int_0^1 ds(X)\frac{\partial}{\partial s(X)}f(\{s\}_{T(\Lambda)})$$

and so

$$\begin{aligned}&\prod_{i=1}^m \delta^{X_i} f(\{s\}_{T(\Lambda)-\{X_i\}_1^m},\ \{0\}_{\{X_i\}_1^m})\\ &=\int_0^1\cdots\int_0^1\prod^m ds(X_i)\prod_{i=1}^m\frac{\partial}{\partial s(X_i)}f(\{s\}_{T(\Lambda)})\end{aligned}\tag{5.2.1}$$

where we have used the definitions in (5.1.7). From the definitions in (5.1.3) and (5.1.6) and from (5.2.1) it follows that

$$\begin{aligned}&|\prod_{i=1}^m \delta^{X_i} f_{B\cup Y}(\{0\}_{T(B\cup Y)})|\\ &\leq\int d\mu_{B\cup Y}[\int_0^1 ds(X_1)\ldots\int_0^1 ds(X_n)\prod_{i=1}^m|W(\omega_{X_i})|\\ &\quad\cdot e^{i\phi(f)}\exp[-U(\omega_{B\cup Y},\ \{s\}_{\{X_i\}},\ \{0\}_{B\setminus Y})]\end{aligned}\tag{5.2.2}$$

where $Y=UX_i$ and $W(\omega_{X_i})=W(\omega_j,\omega_l)$ if $X_i=\{j,l\}$.

We now use Assumption 2.3 (b) to get that for a given $\gamma>0$

$$\begin{aligned}&\prod_{X_i}|W(\omega_{X_i})|\\ &\leq\prod_{X_i}(\beta^\gamma\phi(d(X_i))^{1/2})\prod_{\substack{X_i\\ X_i=\{j,l\}}}\beta^{-\gamma}\frac{1}{2}\phi(|j-l|)^{1/2}(\omega_j^2+\omega_l^2)\\ &\leq\prod_{X_i}(\beta^\gamma\phi(d(X_i))^{1/2}\prod_{j\in Y}(\beta^{-\gamma}A'\omega_j^2)\end{aligned}\tag{5.2.3}$$

where $d(X_i)=|j-k|$ if $X_i=\{j,k\}$ and $A'=\sum_{j\in Z^\nu}\phi(|j|)^{1/2}$.

Assumption 2.3 and (3.3.18) give the bound

$$-U(\omega_{B\cup Y},\ \{s\}_{\{X_j\}},\ \{0\}_{B\setminus Y})+i\phi(f)\tag{5.2.4}$$

$$\leq -\sum_{i\in B\setminus Y} U(\omega_i) - A''\sum_{i\in Y} U(\omega_i) + c(|||f|||+|B|)$$

for some constants A'' and c. We use (5. 2. 3) and (5. 2. 4) to conclude that for $0<\gamma<(\alpha-2)/\alpha$

$$(5.2.2) \leq (Z_\beta^{(0)})^{|B\setminus Y|} \left(\prod_{i\in Y} \beta^{-\gamma} A'' \int d\mu(x_i, \omega_i)\omega_i^2 \exp[-A''U(\omega_i)]\right) \cdot \left(\prod_{X_i} \beta^\gamma \phi(d(X_i))^{1/2}\right) e^{c(|||f^-|||+|B|)}. \qquad (5.2.5)$$

Combining (5. 1. 8), (5. 2. 5) and Lemma 5. 2. 3 we obtain the bound

$$|K^\beta(B, Y, f_{B\cup Y})| \leq e^{c(|||f|||+|B|)} \sum_{\substack{Y\subset \Lambda \\ Y\cap B\neq \phi}} e^{c\beta|B\cup Y|}\ F^\beta(B, Y; \Psi) \qquad (5.2.6)$$

where

$$F^\beta(B, Y; \phi) = \sum_{\substack{\{X_1,\dots,X_n\}\subset T(\Lambda) \\ \cup X_i = Y \\ \{B, X_1, \dots, X_n\} \text{ connected}}} \prod_{i=1}^{n} (\beta^\gamma \phi(d(X_i))^{1/2}). \qquad (5.2.7)$$

The expression (5. 2. 6) and (5. 2. 7) correspond to the expressions (3. 7) and (3. 6) of [9] respectively. We write

$$f_\beta(X) = \beta^\gamma \phi(d(X))^{1/2}.$$

By Assumption 2. 3 we have that for fixed i and $X=\{i, j\}$

$$\sum_{\substack{X\subset Z^\nu \\ \text{card}(X)=2}} \phi(d(X))^{1/2} < \infty. \qquad (5.2.8)$$

What we have done until now in this section is the derivation of the bound (5. 2. 6) which ensures the convergence of the cluster expansion, provided (5. 2. 8) holds. In more explicit, we replace $F^\beta(X_0, X; \Phi)$ in (3. 7) of [9] by $F^\beta(B, Y; \phi)$ in (5. 2. 7) and follow each step used in the below of (3. 7) of [9]. Then the bound in (5. 2. 8) guarantees the convergence of the cluster expansion. For details we refer to [9].

Proof of Theorem 2. 5 (a). Theorem 5. 3. 1 and Lemma 5. 3. 2 (b) implies the first part of the theorem. The proof of the cluster property from the convergence of the cluster expansion is a standard method in statistical mechanics. For an instance, see [17].

Sketch of Proof of Lemma 5.2.2. We assume that Lemma 5.2.3 holds. Following the method used to obtain (5.2.6) one may obtain the bound of the form

$$|K^{\beta}(\{i_0\}, S, \hat{Z}_{\{i_0\}\cup Y})| \leq e^{c|S|} F^{\beta}(\{i_0\}, S; \phi). \tag{5.2.9}$$

Using the above bound and following the method used in the proof of proposition 4.4 of [9] one obtains that for small $\beta < \beta_0$

$$\|\chi_{\Lambda} K \chi_{\Lambda}\| < 1 \text{ uniformly in } \Lambda.$$

As in page 570 of [9], the above implies Lemma 5.2.2. For the details, see section 4 of [9].

We now come to the proof of Lemma 5.2.3 which makes the completion of the proof of Theorem 2.5(a).

Proof of Lemma 5.2.3. We first assert that there are constants A_1 and c_1 independent of β such that

$$Z_{\beta}^{(0)} \geq A_1 e^{-c_1\beta} \beta^{-d[(1/2)+(1/\alpha)]}. \tag{5.2.10}$$

Let $\Delta(j)$, $j \in Z^d$, be the unit cubes in R^d centered at j, and let $U(j) = \sup_{y \in \Delta(j)} U(y)$. Then a direct calculation gives

$$\begin{aligned} Z_{\beta}^{(0)} &\geq \int d\mu(x; \omega) e^{-U(\omega)} [\sum_{j \in Z^d} \chi_{\Delta(j)}(\omega)] \\ &\geq [\int d\mu(x; \omega) \chi_{\Delta(0)}(\omega)] (\sum_{j \in Z^d} \exp(-\beta U(j))) \\ &\geq [A_1' \beta^{-d/2}] (A_1'' \; e^{-d/\alpha} e^{c_1\beta}). \end{aligned}$$

This proves (5.2.10). On the other hand, let $H = -\frac{1}{2}\Delta + A' U(x)$. Then

$$\begin{aligned} &\int d\mu(x, \omega) \omega^2 \exp[-A' U(\omega)] \\ &\qquad = \beta \, Tr(x^2 e^{-\beta H}) \\ &\qquad \leq \beta [Tr(|x|^{\alpha} e^{-\beta H})]^{2/\alpha} [Tr(e^{-\beta H})]^{1-(2/\alpha)} \\ &\qquad \equiv \beta D_1(\beta)^{2/\alpha} D_2(\beta)^{1-(2/\alpha)}. \end{aligned} \tag{5.2.11}$$

Using Golden–Thompson inequality [Theorem 9.2. of [15]] and Assumption 2.3 (a) it is easy to show that

$$D_2(\beta) \leq A_2 e^{\beta c} \beta^{-d(1/2+1/\alpha)}. \tag{5.2.12}$$

Since $|x|^\alpha \leq D(U(x)+1) \leq D'H$ by Assumption 2.3 (a) for some constant $D'>0$, it follows that

$$\begin{aligned} D_1(\beta) &\leq \| |x|^\alpha \exp[-\frac{1}{2}\beta H\| \; Tr(\exp[-\frac{1}{2}\beta H]) \\ &\leq (A_2'\beta^{-1}e^{\beta c'})(A_2''e^{\beta c''}\beta^{-d(1/2+1/\alpha)}). \end{aligned} \tag{5.2.13}$$

Combining (5.2.11) − (5.2.13) we have the bound

$$(5.2.11) \leq De^{\beta d}\beta^{-d(1/2+1/\alpha)}. \tag{5.2.14}$$

The lemma now follows from (5.2.10), (5.2.11) and (5.2.14). This proves the lemma completely.

6. Discussions

We first discuss the construction approachs of infinite volume physical Hilbert spaces, unitary representations of time evolutions and cyclic vectors, which we have used in Section 2. In the first approach we started from the quasi-local algebra $\mathcal{A}$ of local observables and finite volume Gibbs sttes $\omega_{\Lambda,\beta}$ to construct $(\mathcal{H}, U, \Omega)$ [Theorem 2.3]. In the second approach we started from interacting measures $dp_{\Lambda,\beta}$ and followed the standard method in constructive quantum field theory to construct $(\hat{\mathcal{H}}, \hat{U}, \hat{\Omega})$ [Theorem 2.4 and Theorem 2.5]. We would like to know that the precise relationship between $(\mathcal{H}, U, \Omega)$ and $(\hat{\mathcal{H}}, \hat{U}, \hat{\Omega})$ is. In quantum field theory the two approachs seem to give the same theory because of locality of interactions. In our case the interactions $U(x_\Lambda)$ are not local.

Finally, we give a few comments on phase transitions and continuous symmetric breaking. Let us assume the interaction is of the form

$$U(x_\Lambda) = \sum_{i\in\Lambda} U_1(|x_i|^2) + \sum_{i,j\in\Lambda} J(|i-j|)x_i \cdot x_i.$$

That is. the interaction is rotation invariant. If $J(|i-j|)$ satisfies some additional properties such as ferromagnetic type nearest interaction, then for $\nu \geq 3$ it may be possible to show the existence of a continuous symmetric breaking for sufficiently large β [4]. If $\nu \leq 2$, one may expect the absence of continuous symmetric breaking [5]. However, we do not try to check those here.

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