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THE WEIGHTED GENERALIZED INVERSE OF A LINEAR OPERATOR AND REGULARIZATION

MAN-SUK SONG

1. Introduction

In this paper we introduce the weighted generalized inverse of a linear operator in a general Hilbert space setting and the method of regularization to obtain an approximate solution to an ill-posed constrained minimization problem.

Let X and Y be (real or complex) spaces with $\langle .,. \rangle$, $\|\cdot\|$ denoting the inner product and the norm for both. Let A be a bounded linear operator from X into Y, and let A^* denote the adjoint of A, i. e., for all $x \in X$, $y \in Y$,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle.$$

Let $R(A)$ and $N(A)$ denote the range and the null space of A respectively. For any subspace S, we denote the orthogonal complement of S by $S^\perp$ and the closure of S by $\bar{S}$. It is well known [9] that

$$X = N(A) \oplus N(A)^\perp$$
$$Y = N(A^*) \oplus N(A^*)^\perp$$
$$\{\overline{R(A)}\}^\perp = N(A^*), \quad \overline{R(A^*)} = N(A)^\perp$$

For a given $b \in Y$, an element $u \in X$ is called a *least squares solution* of the operator equation

$$(1) \qquad Ax = b$$

if $\|Au - b\| \leqq \|Ax - b\|$ for all $x \in X$.

Among least squares solutions an element $\bar{u}$ of minimal norm is called a pseudosolution or a best approximate solution of (1), i. e., $\|\bar{u}\| \leqq \|u\|$ for all least squares solutions u. For each $b \in R(A) + R(A)^\perp$, the set of all least squares solutions of (1) is non-empty, closed, and convex. Hence it has a unique element $\bar{u}$ of minimal norm. The *generalized*

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inverse of A, denoted by $A^\dagger$, is defined as the operator which assigns to each $b \in R(A)+R(A)^\perp$, the unique least squares solution of minimal norm of the equation (1). Then $A^\dagger$ is a linear operator from $R(A)+R(A)^\perp$ into X. If $R(A)$ is closed, the domain of $A^\dagger, D(A^\dagger)$, becomes the whole space Y and $A^\dagger$ is bounded. But $A^\dagger$ is unbounded and $R(A)+R(A)^\perp \neq Y$ when $R(A)$ is not closed.

The operator equation (1) is said to be *well-posed*(relative to the spaces X and Y) if for each $y \in Y$ it has a unique pseudosolution which depends continuously on y; otherwise the equation is said to be *ill-posed.*

The following statements are equivalent [6]:

(a) The operator equation (1) is well-posed in (X, Y).
(b) A has a colsed range in Y.
(c) A^+ is a bounded linear operator on Y into X.

Let L be a bounded linear operator from X into Z, where Z is a third Hilbert space. We assume that the range of L, $R(L)$, is closed in Z, but the range of A, $R(A)$, is not necessarily closed in Y. For a given y in the domain of $A^\dagger$, let

$$S_y = \{u \in X : \|Au-y\|_Y = \inf \|Ax-y\|_Y, x \in X\}. \tag{2}$$

Then we consider the following minimization problem: Find an element $w \in S_y$ such that

$$\|Lw\|_Z = \inf \ \{\|Lu\|_Z : u \in S_y\}. \tag{3}$$

This problem (2)-(3) is generally known as a constrained minimization problem with constraint operator L.

Since for any $u \in S_y$, $u = A^\dagger y + v$ for fome $v \in N(A)$, the problem(3) is equivalent to

$$\inf_{u \in S_y} \|Lu\| = \inf_{v \in N(A)} \|L(A^+ y + v)\| = \inf_{u \in LS_y} \|u\|.$$

Noting that LS_y is a translate of the subspace $LN(A)$, the problem (2)-(3) has a solution for every $y \in D(A^\dagger)$ if and only if $LN(A)$ is closed, and the solution is unique if and only if $N(A) \cap N(L) = \{0\}$ [6]. It is not difficult to show that $LN(A)$ is closed if and only if $N(A)+N(L)$ is closed. Throughout this paper, we assume that $N(A) \cap N(L) = \{0\}$ and $N(A)+N(L)$ is closed, i. e., that the constrained minimization problem (2)-(3) has a solution for each $y \in D(A^\dagger)$ and the solution is unique.

2. Weighted generalized inverse

Let

(4) $$[u, v] := \langle Au, Av\rangle_Y + \langle Lu, Lv\rangle_Z \text{ for } u, v \in X$$

and

$$M = \{x \in X : L^*Lx \in N(A)^\perp\}.$$

Then the following proposition is an immediate consequence of the definition of $[.,.]$ and the assumption that $N(A) \cap N(L) = \{0\}$.

PROPOSITION 1. (a) (4) *defines an inner product in* X.
(b) M *is a closed subspace of* X *and is orthogonal complement of* $N(A)$ *with respect to the new inner product* (4), *i.e.*,

$$X = N(A) \oplus_L M$$

where $\oplus_L$ *denotes the orthogonal decomposition with respect to* $[.,.]$.

Let X_L denote the space X with the new inner product $[.,.]$.

THEOREM 2. *An element* $w \in X$ *is a solution to the problem* (2)-(3) *if and only if* $A^*Aw = A^*y$ *and* $L^*Lw \in N(A)^\perp$.

Proof. See, for example, Nashed [6].

According to the Theorem 2, the problem of constrained minimlzation problem (2)-(3) is equivalent to finding an element $w \in M$ such that $A^*Aw = A^*y$. Thus the solution w is the least squares solution of X_L-minimal norm of the equation (1). Let $A_L{}^\dagger$ denote the map induced by $y \to w$ and call it *the weighted generalized inverse of* A. Then the weighted generalized inverse $A_L{}^\dagger$ is the generalized inverse of A relative to the decompositions

$$X = N(A) \oplus_L M \text{ and } Y = \overline{R(A)} \oplus R(A)^\perp.$$

The relation between $A^\dagger$ and $A_L{}^\dagger$ is

$$A_L{}^\dagger = (2I - A^\dagger A - U)A^\dagger$$

where U is the projector of X onto $N(A)$ along M [6], [8].

3. Regularization. Existence and uniqueness of the regularized solution

When the range of A is closed, the problem (2)-(3) is well posed. Hence our interest is in the case that the range of A is not closed and therefore the problem is ill-posed. As usual for the ill-posed problems, we regularize it by a family of stable minimization problems.

Let W be the product space of Y and Z with the usual inner product:

$$W=Y\times Z,$$
$$\langle (y_1, z_1), (y_2, z_2)\rangle_W=\langle y_1, y_2\rangle_Y+\langle z_1, z_2\rangle_Z$$

for $y_1, y_2\in Y$ and $z_1, z_2\in Z$. From now on we drop the subscripts X, Y and Z for the inner product and norms whenever the meaning is clear from the context.

For $\alpha>0$, let C_α be a linear operator from X into W defined by

$$C_\alpha x=(Ax, \sqrt{\alpha}Lx) \text{ for } x\in X.$$

LEMMA 3. *For $\alpha>0$, the range of C_α, $R(C_\alpha)$, is closed if $R(L)$ and $A(N(L))$ are closed.*

COROLLARY 4. *Suppose that $R(L)$ and $A(N(L))$ are closed, and $N(A)\cap N(L)=\{0\}$. Let $\bar{b}=(y, 0)$ be an element in W. Then for $\alpha>0$, the operator equation $C_\alpha x=\bar{b}$ is well posed.*

By the Corollary 4, the operator C_α has the bounded generalized inverse C_α^+ defined on all of W for $\alpha>0$ when $A(N(L))$ is assumed to be closed in addition. For the given $\bar{b}\in W$, let U_α denote the unique least squares solution of minimal norm of the equation $C_\alpha x=\bar{b}$ for each $\alpha>0$ i.e., $U_\alpha=C_\alpha^+\bar{b}$.

From the definition of C_α and inner product of W, we get

$$C_\alpha x-\bar{b}=(Ax-y, \sqrt{\alpha}Lx)$$

and

$$\|C_\alpha x-\bar{b}\|^2=\|Ax-y\|^2+\alpha\|Lx\|^2. \tag{5}$$

Let us write

$$J_\alpha(x)=\|Ax-y\|^2+\alpha\|Lx\|^2. \tag{6}$$

THEOREM 5. *Let $\alpha>0$. An element x_α in X minimizes the quadratic functional $J_\alpha(x)$ if and only if*

$$(A^*A+\alpha L^*L)x_\alpha=A^*y. \tag{7}$$

Proof. It is easy and is omitted.

By Theorem 5 and (5), we conclude that $U_\alpha=C_\alpha{}^+\bar{b}$ is the unique minimizer of the quadratic functional $J_\alpha(x)$ and also the unique solution of the equation (7), which is equivalent to $C_\alpha{}^*C_\alpha x=C_\alpha{}^*\bar{b}$. *Throughout the paper we assume that $AN(L)$ is closed.* Then $R(C_\alpha)$ is closed for $\alpha>0$ and in particular $R(C_1)$ is closed. Therefore C_1 has a bounded inverse on $R(C_1)$ and hence there exist constants m_1 and m_2 such that $0<m_1\leqq m_2<\infty$ and

$$m_1\|x\|\leqq\|C_1x\|\leqq m_2\|x\| \text{ for all } x\in X. \tag{8}$$

We denote the new norm derived by $\|C_1x\|$ by $\|x\|_L$ i. e., $\|x\|_L{}^2=\|Ax\|^2+\|Lx\|^2$.

PROPOSITION 6. *Suppose that $R(L)$ and $LN(A)$ are closed.*

(a) *If $A(M)$ is closed, then $R(C_\alpha)$ is closed.*
(b) *If $R(A)$ is closed, then $R(C_\alpha)$ is closed.*
(c) *If $N(L)$ is finite dimensional, then $N(L)+N(A)$ an $R(C_\alpha)$ are closed.*

REMARK. By Lemma 3 and Proposition 6, if either $N(L)$ or M is finite dimensional, then the operator equation (7) is well-posed.

4. Convergence: $\{U_\alpha\}\to A^\dagger{}_L y$.

In this section we show that the regularized solution U_α converges to the solution $A^\dagger{}_L y$ as α tends to 0.

LEMMA 7. *(a) $w\in S_y$ is the solution of the problem (2)-(3) if and only if $w\in M$ and $J_0(w)\leqq J_0(x)$ for all $x\in X$ where $J_0(x)=\|Ax-y\|^2$.*
(b) U_α is the solution of (7) if and only if $J_\alpha(U_\alpha)\leqq J_\alpha(x)$ for all $x\in X$.

LEMMA 8. *Let $\alpha>0$. Then*

$$\|LU_\alpha\|\leqq\|Lw\| \text{ and } \|AU_\alpha\|\leqq\|Aw\|$$

where $w=A^{\dagger}{}_{L}y$.

Proof. Since $J_\alpha(U_\alpha)\leqq J_\alpha(x)$ for any $x\in X$, we get $\|AU_\alpha-y\|^2+\alpha\|LU_\alpha\|^2\leqq\|Aw-y\|^2+\alpha\|Lw\|^2$. By Lemma 7-(a), $\|Aw-y\|^2\leqq\|AU_\alpha-y\|^2$ and we get the first inequality. Since $A^*Aw=A^*y$, $\langle A^*Aw, w\rangle=\langle A^*y, w\rangle$. Similarly, $A^*AU_\alpha+\alpha L^*LU_\alpha=A^*y$ implies $\langle A^*AU_\alpha, U_\alpha\rangle+\alpha\langle L^*LU_\alpha, U_\alpha\rangle=\langle A^*Ay, U_\alpha\rangle$. Using these equations, we get

$$\text{(9)}\qquad J_0(w)=\|Aw-y\|^2=\|Aw\|^2-2\langle w, A^*y\rangle+\|y\|^2 \\ =-\|Aw\|^2+\|y\|^2.$$

$$\text{(10)}\qquad J_0(U_\alpha)=\|AU_\alpha-y\|^2=\|AU_\alpha\|^2-2\langle U_\alpha, A^*y\rangle+\|y\|^2 \\ =-\|AU_\alpha\|^2-2\alpha\|LU_\alpha\|^2+\|y\|^2.$$

By the minimizing property of w, $J_0(w)\leqq J_0(U_\alpha)$ which implies $-\|Aw\|^2+\|y\|^2\leqq-\|AU_\alpha\|^2-2\alpha\|LU_\alpha\|^2+\|y\|^2$. Now the second inequality follows from this.

LEMMA 9. *For* $\alpha>0$,
$$\lim_{\alpha\to 0} A^*AU_\alpha=A^*y.$$

LEMMA 10. *The set* $\{U_\alpha\}$ *is bounded in* X *and has a weakly convergent subsequence, say* $\{U_\beta\}$.

LEMMA 11. *Suppose* $\{U_\beta\}$ *converges weakly to* $\bar{w}$. *Then* $\bar{w}\in M$ *and* $\bar{w}=w$.

Proof. For each $\beta>0$, $A^*AU_\beta-A^*y=A^*(AU_\beta-y)=-\beta L^*LU_\beta$. This implies that L^*LU_β belongs to $R(A^*)$ which is contained in $N(A)^\perp$. Hence $\langle v, L^*LU_\beta\rangle=0$ for each $v\in N(A)$. By the weak convergence of $\{U_\beta\}$, we have

$$[v, \bar{w}]=\lim_{\beta\to 0}[v, U_\beta]=\lim_{\beta\to 0}\{\langle A^*Av, U_\beta\rangle+\langle v, L^*LU_\beta\rangle\}=0$$

for each $v\in N(A)$. Therefore $\bar{w}$ is in M. To prove the second statement, we show that $\bar{w}$ minimizes $J_0(x)$ and by uniqueness of w we get the desired result. It suffices to show that $\langle A^*A\bar{w}, x\rangle=\langle A^*y, x\rangle$ for each $x\in X$. For each fixed x, by the weak convergence of $\{U_\beta\}$ and the boundedness of A^*A and L^*L we have

$$\lim_{\beta\to 0}\{\langle A^*AU_\beta, x\rangle+\beta\langle L^*LU_\beta, x\rangle\}=\langle A^*y, x\rangle.$$

Therefore, we get $\langle A^*A\bar{w}, x\rangle=\langle A^*y, x\rangle$.

LEMMA 12. *Suppose* $0<\alpha'<\alpha$. *Then*

$$\|LU_\alpha\| \leqq \|LU_{\alpha'}\|$$

and

$$\|AU_\alpha\|^2 - \|AU_{\alpha'}\|^2 \leqq 2(\alpha\|LU_{\alpha'}\|^2 - \alpha\|LU_\alpha\|^2).$$

LEMMA 13. *Suppose that the subsequence* $\{U_\beta\}$ *converges strongly to* w. *Then* $\{U_\alpha\}$ *converges to* w.

Proof. Since

$$\begin{aligned}\|U_\beta - w\|_L &= (\|AU_\beta - Aw\|^2 + \|LU_\beta - Lw\|^2)^{1/2} \\ &\geqq \left(\frac{1}{2}\right)^{1/2} (\|AU_\beta - Aw\| + \|LU_\beta - Lw\|)\end{aligned}$$

and $\|U_\beta - w\|_L \to 0$ as $\beta \to 0$, we get

$$\|AU_\beta - Aw\| \to 0 \tag{11}$$

$$\|LU_\beta - Lw\| \to 0. \tag{12}$$

By Lemma 11, $\{LU_\alpha\}$ is monotonically increasing sequence as $\alpha \to 0$ and thus by (12), $\{LU_\alpha\}$ converges to Lw. Let $0<\beta<\alpha$. Then in the following inequality

$$\|AU_\alpha - Aw\| \leqq \|AU_\alpha - AU_\beta\| + \|AU_\beta - Aw\|,$$

the second term on 'the right hand side tends to zero by (11). By Lemm 9, $\{AU_\alpha\}$ converges weakly and by Lemma 12, $\{\|AU_\alpha\|\}$ converges to $\|Aw\|$. Hence $\{AU_\alpha\}$ converges to Aw. Thus the first term on the right hand side of the above inequality tends to zero. Therefore

$$\|U_\alpha - w\|_L \leqq \|AU_\alpha - Aw\| + \|LU_\alpha - Lw\| \to 0 \text{ as } \alpha \to 0.$$

By the equivalence of norms, $\{U_\alpha\}$ converges to w in X.

THEOREM 14. *For* $\alpha>0$ *and a given* $y \in Y$,

$$\lim_{\alpha \to 0} U_\alpha = A^\dagger{}_L y.$$

Proof. Since $\{U_\beta\}$ converges to w weakly, we have $\|w\|_L \leqq \lim_{\beta \to 0} \|U_\beta\|_L$. Combining this and Lemma 7, we get $\lim_{\beta \to 0} \|U_\beta - w\|_L = 0$. Then by Lemma 12, $\{U_\alpha\}$ converges to w in X. that is,

$$\lim_{\alpha \to 0} \|U_\alpha - A_L{}^\dagger y\| = 0.$$

We have shown that under the assumption that $N(A)\cap N(L)=\{0\}$, $R(L)$ and $AN(L)$ are closed, the operator equation (7) provides a unique solution U_α which depends continuously on y for each $\alpha>0$, and $U_\alpha \to A_L{}^\dagger y$ as $\alpha\to 0$. Therefore $\{A^*A+\alpha L^*L)^{-1}A^*\}$ is a family of regularizing operators for the constrained minimization problem (2)-(3) in the sense of Tikhonov [10].

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Department of Computer Science
Yonsei University
Seoul 120, Korea